Negative Semidefiniteness, and Concave and Quasiconcave Functions

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1 Semidefiniteness

Definition 1. The $N \times N$ matrix M is **negative semidefinite** (NSD) (**positive semidefinite** (PSD)) if $\forall z \in \mathbb{R}^N$,

$$z \cdot Mz \leq (\geq)\, 0.$$

If the inequality is strict for all $z \neq 0$, then M is **negative definite** (ND) (**positive definite** (PD)).

Example 2. The identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is positive (semi)definite since for all $z = \begin{pmatrix} x \\ y \end{pmatrix}$,

$$z \cdot Iz = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 \geq 0.$$

And $-I$ is negative (semi)definite. The matrix $M = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ is negative semidefinite, as $\forall z = \begin{pmatrix} x \\ y \end{pmatrix}$,

$$z \cdot Mz = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -x-y \\ -x-y \end{pmatrix} = -x^2 - xy - xy - y^2 = -(x+y)^2 \leq 0,$$

but not negative definite at $x = -y$, $z \cdot Mz = 0$. However, not all matrices are either positive or negative semidefinite, for example, $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$z \cdot Dz = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - y^2.$$

Proposition 3. *Some properties of the matrices are*

1. *M is PSD (PD) $\Leftrightarrow -M$ is NSD (ND).*
2. *M is ND (PD) $\Rightarrow M$ is NSD (PSD), but M is NSD (PSD) $\not\Rightarrow M$ is ND (PD).*
3. *M is ND (PD) $\Leftrightarrow M^{-1}$ is ND (PD).*
4. *M is ND (PD) $\Leftrightarrow M + M'$ is ND (PD).*

Proof. Sketches are as follows:

1. $z \cdot (-M)\, z = -(z \cdot Mz)$.

2. For all $z \neq 0$, $z \cdot Mz > 0$ and for $z = 0$, $z \cdot Mz = 0$, then $\forall z, z \cdot Mz \geq 0$.
3. $z \cdot Mz = (z \cdot Mz)' = z \cdot M'z = z \cdot MM^{-1}M'z = M'z \cdot M^{-1}M'z$.
4. $z \cdot (M + M')\, z = 2z \cdot Mz$.

□

2 Concave Functions

Definition 4. On a convex set $A \subset \mathbb{R}^N$, a function $f : A \to \mathbb{R}$ is **concave** (**convex**) if $\forall x, x' \in A$ and $\alpha \in (0, 1]$,

$$f\left(\alpha x' + (1-\alpha) x\right) \geq (\leq)\, \alpha f\left(x'\right) + (1-\alpha) f(x).$$

If the inequality is strict for all $x \neq x'$ and all $\alpha\,(0, 1)$, then the function is **strictly concave** (**strictly convex**).

Proposition 5. *Equivalently, a function is concave if* $\forall x_1, \cdots, x_k \in A$*,*

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) \geq \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$$

such that $\alpha_1 + \cdots + \alpha_k = 1$.

Proof. Since A is convex, if $f(\alpha_1 x_1 + (1-\alpha_1) x_1') \geq \alpha_1 f(x_1) + (1-\alpha_1) f(x_1')$,

$$\begin{aligned} & f\left(\alpha_1 x_1 + (1-\alpha_1)\left(\frac{\alpha_2}{1-\alpha_1} x_2 + \cdots + \frac{\alpha_k}{1-\alpha_1} x_k\right)\right) \\ \geq\ & \alpha_1 f(x_1) + (1-\alpha_1) f\left(\frac{\alpha_2}{1-\alpha_1} x_2 + \cdots + \frac{\alpha_k}{1-\alpha_1} x_k\right) \\ \geq\ & \alpha_1 f(x_1) + (1-\alpha_1) \frac{\alpha_2}{1-\alpha_1} f(x_2) + (1-\alpha_1-\alpha_2) f\left(\frac{\alpha_3}{1-\alpha_2} x_3 + \cdots + \frac{\alpha_k}{1-\alpha_2} x_k\right) \end{aligned}$$

□

In essence, $f(\sum \alpha_i x_i) \geq \sum \alpha_i f(x_i)$ with $\sum_i \alpha_i = 1$. If we think α_i as probability weight, in general,

Proposition 6 (Jensen's Inequality)**.** *If* $f : \mathbb{R} \to \mathbb{R}$ *is concave,* $f\left(\int x dF\right) \geq \int f(x)\, dF$.

The second characterization of concave function is by the condition that any tangent to the graph of a concave function must lie (weakly) above the graph of $f(\cdot)$.

Proposition 7. *The (continuously differentiable) function* $f : A \to \mathbb{R}$ *is concave if and only if*

$$f(x+z) \leq f(x) + \nabla f(x) \cdot z$$

for all $x \in A$ *and* $z \in \mathbb{R}^N$ *(with* $x + z \in A$*).*

Proof. Take x and $x + z$, for given α,

$$\begin{aligned} f(\alpha(x+z) + (1-\alpha) x) &\geq \alpha f(x+z) + (1-\alpha) f(x) \\ f(x+\alpha z) - f(x) &\geq \alpha (f(x+z) - f(x)) \\ f(x) + \frac{f(x+\alpha z) - f(x)}{\alpha} &\geq f(x+z) \end{aligned}$$

Take $\alpha \to 0$, then we have the desired inequality. □

3 Connections between Concavity and NSD

Proposition 8. *The (twice continuously differentiable) function $f : A \to \mathbb{R}$ is concave if and only if $D^2 f(x)$ is NSD for every $x \in A$. If $D^2 f(x)$ is ND, then the function is strictly concave.*

Proof. We first show that concavity implies Hessian matrix is NSD. Suppose f is concave. Fix some $x \in A$, with some $z \neq 0$, take second-order Taylor expansion,

$$f(x+\alpha z) = f(x) + \nabla f(x) \cdot (\alpha z) + \frac{\alpha^2}{2} z \cdot D^2 f(x+\beta z) z. \tag{1}$$

By Proposition 7,

$$\frac{\alpha^2}{2} z \cdot D^2 f(x+\beta z) z \leq 0.$$

Take α arbitrarily small, then β is arbitrarily small, so

$$z \cdot D^2 f(x) z \leq 0. \tag{2}$$

And we can show sufficiency by the same equation 1 coupled with the condition in equation 2. □

Proposition 9 (Checking for ND and NSD)**.** *Let M be a symmetric $N \times N$ matrix.*

1. *M is ND if and only if $(-1)^r \det({}_rM_r) > 0$ for every $r = 1, \cdots, N$, where ${}_tM_s$ denotes the matrix of M with the first t rows and first s columns.*
2. *M is NSD if and only if $(-1)^r \det({}_rM_r) \geq 0$ for every $r = 1, \cdots, N$, and for every permutation π of the indices $\{1, \cdots, N\}$.*
3. *M is PD if and only if $\det({}_rM_r) > 0$ for every $r = 1, \cdots, N$.*
4. *M is PSD if and only if $\det({}_rM_r) \geq 0$ for every $r = 1, \cdots, N$, and for every permutation π of the indices $\{1, \cdots, N\}$.*

Example 10. Check if $f(x_1, x_2)$ is concave by checking the definiteness of the Hessian matrix,

$$D^2 f(x_1, x_2) = \begin{bmatrix} f_{11}(x_1, x_2) & f_{12}(x_1, x_2) \\ f_{21}(x_1, x_2) & f_{22}(x_1, x_2) \end{bmatrix}.$$

To see if the function is strictly concave, we check for ND, and we need that, by Proposition 9,

$$\det(f_{11}) < 0, \det D^2 f(x_1, x_2) > 0,$$

or $f_{11} < 0$ and $f_{11} f_{22} - f_{12}^2 > 0$.

To check for concavity of the function, we check for NSD of Hessian matrix, i.e.,

$$\det(f_{11}) \leq 0, \det D^2 f(x_1, x_2) \geq 0,$$

and permutating,

$$\det(f_{22}) \leq 0, \det \begin{vmatrix} f_{22} & f_{21} \\ f_{12} & f_{11} \end{vmatrix} \geq 0.$$

In summary, check $f_{11}, f_{22}, f_{12}^2 - f_{11} f_{12} \leq 0$.

4 Quasiconcave Functions

Definition 11. The function $f : A \to \mathbb{R}$ is **quasiconcave** (**quasiconvex**) if its upper (lower) contour sets $\{x \in A : f\{x\} \geq t\}$ are convex sets; that is, if $\forall t \in \mathbb{R}$, $x, x' \in A$, and $\alpha \in [0,1]$

$$f(x), f(x') \geq (\leq) t \Rightarrow f(\alpha x + (1-\alpha) x') \geq (\leq) t.$$

If the inequality is strict whenever $x \neq x'$ and $\alpha \in (0,1)$, then f is **strictly quasiconcave** (**strictly quasiconvex**).

Remark 12. It follows that $f(\cdot)$ is quasiconcave if and only if $\forall x, x' \in A$ and $\alpha \in [0,1]$,

$$f(\alpha x + (1-\alpha) x') \geq \min\{f(x), f(x')\}.$$

Thus, a concave function is automatically a quasiconcave function. However, the converse is not true: for example, all increasing functions of one variable are quasiconcave but they are not necessarily concave.

More importantly, a concave function is not preserved under an increasing transformation of $f(\cdot)$, for example, $(\sqrt{x})^4$. However, quasiconcavity is preserved under the transformation. Therefore, concavity is a "cardinal" property, but quasiconcavity is "ordinal" property.

Proposition 13. *The (twice continuously differentiable) function $f : A \to \mathbb{R}$ is quasiconcave if and only if for every $x \in A$, the Hessian matrix $D^2 f(x)$ is NSD in the subspace $\{z \in \mathbb{R}^N : \nabla f(x) \cdot z = 0\}$.*

The proof is the same to that of Proposition 8 and follows from the following lemma.

Lemma 14. *The (continuously differentiable) function $f : A \to \mathbb{R}$ is quasiconcave if and only if $\forall x, x' \in A$ such that $f(x') \geq f(x)$,*

$$\nabla f(x) \cdot (x' - x) \geq 0.$$

5 Extended Reading

The notes are extracted from pps 930-940, Mas-Colell, Whinston, and Green (1995). For an equally superior and more detailed treatment of the materials, please refer to Chapter 21, pp 505-543 of Simon and Blume (1994) on concave and quasiconcave functions, and to Chapter 16, pp 375-393 of Simon and Blume (1994) for detailed discussions of semidefinite matrices in optimization.

MAS-COLELL, A., M. D. WHINSTON, AND J. R. GREEN (1995): *Microeconomic Theory*. Oxford University Press.

SIMON, C., AND L. BLUME (1994): *Mathematics for Economists*. WW Norton New York.