Generic Steps in Solving Indirect Selling Mechanisms

Hanzhe Zhang*

Friday, January 27, 2012†

In this note, we explore the generic steps of solving problems of indirect selling mechanisms including First Price Auction (FPA), Second Price Auction (SPA), All-Pay First Price Auction (APFPA), All-Pay Second Price Auction (APSPA)[1]. In general, given the rules of an auction (based on bids, how allocation of objects is determined and how payments are determined), steps are

1. State expected utility of a bidder given values and bids
2. Solve the equilibrium bidding function by First Order Condition
3. Compute the revenue

First, because every bidder has ex ante the same value distribution and bidders with higher values will bid higher, we assume that bidding functions are **symmetric** and **increasing**.

1 Expected Utility

Utility when i plays according to r_i conditional on the value vector (v_i, v_{-i}) is $\underline{u_i\left(r_i|v_i, v_{-i}\right)}$, and then expected utility can be interprted in several ways as follows,

$$\begin{aligned}
& u_i\left(r_i|v_i\right) \\
= & \int\int\int_{v_{-i}} u_i\left(r_i|v_i, v_{-i}\right) dF_{-i}\left(v_{-i}\right) \\
= & \Pr\left(\text{win}\right)\left(v_i - \text{winning payment}\right) + \Pr\left(\text{lose}\right)\left(0 - \text{losing payment}\right) \\
= & \Pr\left(\text{win}\right) v_i - \left[\Pr\left(\text{win}\right)\left(\text{winning payment}\right) + \Pr\left(\text{lose}\right)\left(\text{losing payment}\right)\right] \\
= & \text{expected utility of object} - \left[\text{expected payment}\right] \\
= & p_i\left(r_i, v_{-i}\right) v_i - p_i\left(r_i, v_{-i}\right) \text{winning payment} - \left(1 - p_i\left(r_i, v_{-i}\right)\right) \text{losing payment}
\end{aligned}$$

*hanzhe@uchicago.edu. All errors are solely mine.

†Updated 01:00, 01/31/2012.

[1]Recall that SPA is strategically equivalent to English (ascending) auction, and FPA to Dutch (descending) auction.

Then when auction is efficient (i.e., assigning the object to the highest bidder),

$$u_i\left(r_i|v_i\right) = F^{N-1}\left(r_i\right)v_i - \mathrm{WP}\left(r_i, v_{-i}\right) - \left(1 - F^{N-1}\left(r_i\right)\right)\mathrm{LP}\left(r_i, v_{-i}\right).$$

In standard auctions, LP$= 0$, and in all-pay auctions, WP> 0.

- FPA: $u_i\left(r_i|v_i\right) = F^{N-1}\left(r_i\right)\left(v_i - b\left(r_i\right)\right)$
- SPA:

$$\begin{aligned}
u_i\left(r_i|v_i\right) &= F^{N-1}\left(r_i\right)\left(v_i - \mathbb{E}\left[\max_{j\neq i} b\left(v_j\right)|v_j < r_i \forall j \neq i\right]\right) \\
&= F^{N-1}\left(r_i\right)\left(v_i - \frac{1}{F^{N-1}\left(r_i\right)}\int_0^{r_i} b\left(x\right)F^{N-2}\left(x\right)\left(N-1\right)f\left(x\right)dx\right) \\
&= F^{N-1}\left(r_i\right)v_i - \int_0^{r_i} b\left(x\right)F^{N-2}\left(x\right)\left(N-1\right)f\left(x\right)dx
\end{aligned}$$

- APFPA: $u_i\left(r_i|v_i\right) = F^{N-1}\left(r_i\right)v_i - b\left(r_i\right)$
- APSPA (with N bidders instead of just two bidders):

$$\begin{aligned}
u_i\left(r_i|v_i\right) &= \Pr\left(\text{win}\right)\left(v_i - \text{winning payment}\right) + \Pr\left(\text{lose}\right)\left(0 - \text{losing payment}\right) \\
&= \int_{r_i > v_j \forall j \neq i}\left(v_i - b\left(\max_{j\neq i} v_j\right)\right)dF_{-i}\left(v_{-i}\right) + \int_{\exists j: r_i < v_j}\left(-b\left(r_i\right)\right) \\
&= \int_{r_i > \max_{j\neq i} v_j}\left(v_i - b\left(\max_{j\neq i} v_j\right)\right)dF_{-i}\left(v_{-i}\right) + \int_{r_i < \max_{j\neq i} v_j}\left(-b\left(r_i\right)\right) \\
&= \int_0^{r_i}\left(v_i - b\left(x\right)\right)dF^{N-1}\left(x\right) + \int_{r_i}^{1}\left(-b\left(r_i\right)\right)dF^{N-1}\left(x\right) \\
&= \int_0^{r_i}\left(v - b\left(x\right)\right)dF^{N-1}\left(x\right) + b\left(r_i\right)\left(F^{N-1}\left(r_i\right) - 1\right)
\end{aligned}$$

2 Equilibrium Bidding Function

Equilibrium bidding function is derived based on the fact that expected utility is maximized when agent reports truthfully when others report truthfully (incentive-compatibility), i.e., differentiate the expected utility function with respect to r_i and set to 0 (first order condition),

$$\left.\frac{\partial u_i\left(r_i|v_i\right)}{\partial r_i}\right|_{r_i = v_i} = 0.$$

We show the derivation for APSPA with N bidders as well as SPA.

- APSPA

$$\begin{aligned}\frac{\partial u_i(r_i|v_i)}{\partial r_i} &= (v_i - b(r_i))\,F^{N-2}(r_i)(N-1)f(r_i) + b'(r_i)\left(F^{N-1}(r_i)-1\right)\\ &\quad + b(r_i)(N-1)F^{N-2}(r_i)f(r_i)\\ \left.\frac{\partial u_i(r_i|v_i)}{\partial r_i}\right|_{r_i=v_i} &= v_iF^{N-2}(v_i)(N-1)f(v_i) + b'(v_i)\left(F^{N-1}(r_i)-1\right) = 0\end{aligned}$$

Rearrange,

$$b'(v_i) = \frac{F^{N-2}(v_i)(N-1)f(v_i)}{1-F^{N-1}(v_i)}$$

Coupled with the boundary condition $b(0) = 0$, the equilibrium bidding function is

$$b(v_i) = \int_0^{v_i} \frac{(N-1)F^{N-2}(x)\,x}{1-F^{N-1}(x)} f(x)\,dx.$$

When $N = 2$, we get exactly the answer in Pset 1 Problem 5,

$$b(v_i) = \int_0^{v_i} \frac{f(x)\,x}{1-F(x)}dx$$

- SPA: The argument that $b(v) = v$ is without assumption on value distributions of the bidders, here we show in particular that when bidders' values are ex ante the same, equilibrium bidding function is v.

$$\begin{aligned}\frac{\partial u_i(r_i|v_i)}{\partial r_i} &= (N-1)F^{N-2}(r_i)\,v_if(r_i) - b(r_i)F^{N-2}(r_i)(N-1)f(r_i)\\ \left.\frac{\partial u_i(r_i|v_i)}{\partial r_i}\right|_{r_i=v_i} &= (N-1)F^{N-2}(v_i)f(r_i)(v_i - b(v_i)) = 0\end{aligned}$$

Therefore $b(v) = v$.

For derivation of equilibrium bidding function in FPA and APFPA, refer to class notes and Pset 1 Problem 4, respectively, here we summarize the interpretations.

- FPA

$$\begin{aligned}b(v) &= \frac{1}{F^{N-1}(v)}\int_0^v x\,dF^{N-1}(x)\\ &= \frac{1}{G(v)}\int_0^v x\,dG(x) \qquad \left(G(v)\equiv F^{N-1}(v)\right)\\ &= \mathbb{E}\left[\max_{j\neq i} v_j | v_j < v\ \forall j\neq i\right]\\ &= v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{N-1} dx\end{aligned}$$

where the last term denotes how much the bidder *shades* his bid (Pset 1 Problem 2).

- APFPA

$$b(v) = \int_0^v x dF^{N-1}(v) = F^{N-1}(v) \cdot b^{\text{FPA}}(v)$$

In fact, taking the derivative of the expected utility with respect to r_i and setting it equal to 0 to solve for a bidding function only shows that IF a symmetric and increasing equilibrium exists, then the equilibrium bidding function takes the form solved. In order to be sure that such equilibrium exists, we need to show that

1. the utility is actually maximized at $r_i = v_i$ and
2. the utility at $r_i = v_i$ is non-negative (individual rationality holds)

Mathematically, we need to check:

1. $\left.\frac{\partial^2 u_i(r_i|v_i)}{\partial r_i}\right|_{r_i=v_i} < 0.$
2. $u_i(r_i|v_i) \geq 0.$

For a detailed example, please refer to Expanded Solution to Pset 5 Question 5.

3 Revenue Comparisons and Direct Selling Mechanisms

Given the equilibrium bid function, the auctioneer can calculate the expected revenue. We directly calculated those for FPA and SPA in class (by exchanging the double integral signs), and APFPA and APSPA with 2 bidders in Pset 1 (mathematical techniques presented in TA Session 2).

Alternatively, we showed that these auctions are incentive-compatible direct selling mechanisms. According to the Revenue Equivalence Theorem, for any two incentive-compatible direct-selling mechanisms, if

1. Probability assignment function is the same
2. Bidder with value 0 is indifferent between the two mechanisms

then two mechanisms will generate the same expected revenue. Below, we show explicitly that each of the auction is an ICDSM; in particular, note that they are all efficient mechanisms with the same probability assignment function. With $\mathbf{v} = (v_1, \cdots, v_N)$,

- FPA

$$\begin{aligned} p_i(\mathbf{v}) &= 1 \quad v_i > v_j \; \forall j \neq i; 0 \quad \text{otherwise} \\ c_i(\mathbf{v}) &= b^{\text{FPA}}(v_i) \quad v_i > v_j \; \forall j \neq i; 0 \quad \text{otherwise} \end{aligned}$$

- SPA

$$\begin{aligned} p_i(\mathbf{v}) &= 1 \quad v_i > v_j\ \forall j \neq i; 0 \quad \text{otherwise} \\ c_i(\mathbf{v}) &= \max_{j \neq i} b^{\text{SPA}}(v_j) \quad v_i > v_j\ \forall j \neq i; 0 \quad \text{otherwise} \end{aligned}$$

- APFPA

$$\begin{aligned} p_i(\mathbf{v}) &= 1 \quad v_i > v_j\ \forall j \neq i; 0 \quad \text{otherwise} \\ c_i(\mathbf{v}) &= b^{\text{APFPA}}(v_i) \quad v_i > v_j\ \forall j \neq i; 0 \quad \text{otherwise} \end{aligned}$$

- APSPA

$$\begin{aligned} p_i(\mathbf{v}) &= 1 \quad v_i > v_j\ \forall j \neq i; 0 \quad \text{otherwise} \\ c_i(\mathbf{v}) &= \max_{j \neq i} b^{\text{APSPA}}(v_i) \quad v_i > v_j\ \forall j \neq i; 0 \quad \text{otherwise} \end{aligned}$$

ICDSM has the expected cost function fixed up to a constant,

$$\bar{c}_i(v_i) = \bar{c}_i(0) + \bar{p}_i(v_i)\, v_i - \int_0^{v_i} \bar{p}_i(x)\, dx$$

Therefore, if we directly calculate using probability assignment functions, we get the expected cost of bidder i of value i is

$$\begin{aligned} \bar{c}_i(v_i) &= \bar{c}_i(0) + F^{N-1}(v_i)\, v_i - \int_0^{v_i} F^{N-1}(x)\, dx \\ &= F^{N-1}(v_i)\left(v_i - \int_0^{v} \left(\frac{F(x)}{F(v_i)}\right)^{N-1} dx\right) \\ &= b^{\text{APFPA}}(v_i)! \end{aligned}$$

Actually, the result should not be surprising if you think about it: the expected cost when bidder's value is v_i is what he pays under this probability assignment, with the expectation taken over all opposing bidders' distributions; but in APFPA, regardless of other bidders' values, i.e. $\bar{c}_i(v_i) = c_i(v_i, v_{-i})\ \forall v_{-i}$.

These generic steps are useful in understanding how equilibrium bidding functions are derived and how revenues are compared. Of course, they are powerful in solving exam problems, as demonstrated with an example in the TA session.

Coupled with the Mathematical Appendix, this guide should enable you to solve any problem with any auction rules!