Expanded Solutions to Problem Set 1
Problems 1-4

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Question 1. *Show that the symmetric equilibrium bidding function for a first-price auction is strictly increasing.*

Solution. Method 1: Take the derivative with respect to v,

$$\begin{aligned}
\hat{b}'(v) &= -(N-1)F^{-N}(v)f(v)\int_0^v xdF^{N-1}(x) + F^{-N+1}(N-1)vF^{N-2}(v)f(v) \\
&= (N-1)\frac{f(v)}{F(v)}\left[v - F^{-N+1}(v)\int_0^v xdF^{N-1}(x)\right] = (N-1)\frac{f(v)}{F(v)}\left(v-\hat{b}(v)\right)
\end{aligned}$$

Economically $\hat{b}(v) < v$ for all $v > 0$ because all strategies $b(v) > v$ are strictly dominated by $b(v) = v$ because if there is positive probability of winning, the expected payment is bigger than v so the expected payoff is negative if a bidder bids higher than his value. Mathematically or directly from the interpretation of the conditional expected value on winning,

$$F^{-N+1}(v)\int_0^v xdF^{N-1}(x) = \int_0^v xd\left(\frac{F^{N-1}(x)}{F^{N-1}(v)}\right) < \int_0^v vd\left(\frac{F^{N-1}(x)}{F^{N-1}(v)}\right) = v.$$

Method 2: $u(r,v) = F^{N-1}(r)\left(v-\hat{b}(r)\right)$ achieves a critical point r when $r = v$. Consequently,

$$\begin{aligned}
\hat{b}'(v) &= \frac{1}{F^{N-1}(v)}(N-1)f(v)F^{N-2}\left(v-\hat{b}(v)\right) \\
&= \frac{1}{F(v)}(N-1)f(v)\left(v-\hat{b}(v)\right).
\end{aligned}$$

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<u>Method 3</u> (credit to a student): Let $G(v) = F^{N-1}(v)$ and $g(v) = G'(v)$, then

$$\begin{aligned}
\hat{b}(v) &= \frac{1}{G(v)}\int_0^v x dG(x) = \frac{1}{G(v)}\int_0^v x g(x)\,dx \\
\hat{b}'(v) &= -\frac{g(v)}{G^2(v)}\int_0^v x dG(x) + \frac{1}{G(v)} v g(v) \\
&= \frac{g(v)}{G(v)}\left[v - \frac{1}{G(v)}\int_0^v x dG(x)\right] = \frac{g(v)}{G(v)}\left(v - \hat{b}(v)\right)
\end{aligned}$$

Question 2. *Show that the symmetric equilibrium bidding strategy of a first price auction with N symmetric bidders each with values distributed according to F, can be written as*

$$\hat{b}(v) = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{N-1} dx.$$

(Hint: For the first way, use solution from text and apply integration by parts. For the second way, use the fact that $F^{N-1}(r)\left(v - \hat{b}(r)\right)$ is maximized in r when $r = v$ and apply the envelope theorem to conclude that $d\left(F^{N-1}(v)\left(v - \hat{b}(v)\right)\right)/dv = F^{N-1}(v)$; now integrate both sides from 0 to v.)

Solution. <u>Method 1</u>: Integrate by parts, with $u = x$, $du = dx$, and $dv = dF^{N-1}(x)$, $v = F^{N-1}(x)$ in $\int u dv = uv - \int v du$,

$$\begin{aligned}
\hat{b}(v) &= F^{1-N}(v)\left[\left[x \cdot F^{N-1}(x)\right]\Big|_0^v - \int_0^v F^{N-1}(x)\,dx\right] \\
&= F^{1-N}(v)\left[v \cdot F^{N-1}(v) - \int_0^v F^{N-1}(x)\,dx\right] = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{N-1} dx
\end{aligned}$$

<u>Method 2</u>: Since utility is maximized at $r = v$, we have the equality,

$$u\left(\hat{b}(v)\,|v\right) = F^{N-1}(v)\left(v - \hat{b}(v)\right) = \max_{r\in[0,\bar{v}]} F^{N-1}(r)\left(v - \hat{b}(r)\right)$$

By envelope theorem, with respect to v, the equation becomes

$$\begin{aligned}
d\left(F^{N-1}(v)\left(v - \hat{b}(v)\right)\right)/dv &= F^{N-1}(v) \\
d\left(F^{N-1}(x)\left(x - \hat{b}(x)\right)\right) &= F^{N-1}(x)\,dx \\
\int_0^v d\left(F^{N-1}(x)\left(x - \hat{b}(x)\right)\right) &= \int_0^v F^{N-1}(x)\,dx \\
v - \hat{b}(v) &= \frac{1}{F^{N-1}(v)}\int_0^v F^{N-1}(x)\,dx
\end{aligned}$$

Rearrange to get the desired result.

Remark. This form of the equilibrium bidding function highlights the amount by which bidders **shade** their bids (i.e. bid below their value). Note that it is optimal to engage in more shading if there are fewer bidders and less shading as the number of bidders increases. As $N \to \infty$,

$$\hat{b}(v) = v - \int_0^{v-\epsilon/2} \left(\frac{F(x)}{F(v)}\right)^{N-1} dx - \int_{v-\epsilon/2}^{v} \left(\frac{F(x)}{F(v)}\right)^{N-1} dx$$

We have,

$$\begin{aligned}
\left|\int_0^{v-\epsilon/2} \left(\frac{F(x)}{F(v)}\right)^{N-1} dx\right| &\leq \int_0^{v-\epsilon/2} \left|\left(\frac{F(x)}{F(v)}\right)^{N-1}\right| dx \leq \int_0^{v-\epsilon/2} \left|\left(\frac{F\left(v-\frac{\epsilon}{2}\right)}{F(v)}\right)^{N-1}\right| dx \\
\left|\int_{v-\epsilon/2}^{v} \left(\frac{F(x)}{F(v)}\right)^{N-1} dx\right| &\leq \int_{v-\epsilon/2}^{v} \left|\left(\frac{F(x)}{F(v)}\right)^{N-1}\right| dx \leq \int_0^{v-\epsilon/2} 1 dx = \epsilon/2
\end{aligned}$$

Since $\frac{F\left(v-\frac{\epsilon}{2}\right)}{F(v)} < 1$, $\left(\frac{F\left(v-\frac{\epsilon}{2}\right)}{F(v)}\right)^{N-1} \to 0$ as $N \to \infty$ implies that

$$\exists N_0 : \quad \left|\left(\frac{F\left(v-\frac{\epsilon}{2}\right)}{F(v)}\right)^{N-1}\right| \leq \epsilon/2 \ \forall N \geq N_0$$

Then, for all $N \geq N_0$,

$$\left|v - \hat{b}(v)\right| \leq \left|\int_0^{v-\epsilon/2} \left(\frac{F(x)}{F(v)}\right)^{N-1} dx\right| + \left|\int_{v-\epsilon/2}^{v} \left(\frac{F(x)}{F(v)}\right)^{N-1} dx\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Question 3. *This exercise will guide you through the proof that the equilibrium first-price bidding function is in fact a symmetric equilibrium of the first-price auction. (Hint: First show that* $du(r,v)/dr = (N-1)F^{N-2}(r)f(r)(v-r)$ *and conclude that* $du(r,v)/dr$ *is positive when* $r < v$ *and negative when* $r > v$ *so that* $u(r,v)$ *is maximized when* $r = v$*.)*

Solution. Because it holds for all $du(v,v)/dr = 0$, it must hold for $v = r$, that is,

$$0 = du(r,r)/dr = (N-1)F^{N-2}(r)f(r)\left(r - \hat{b}(r)\right) - F^{N-1}(r) \cdot \hat{b}'(r)$$

Then,

$$\begin{aligned}
\frac{du(r,v)}{dr} &= \frac{du(r,v)}{dr} - \frac{du(r,r)}{dr} \\
&= \left[(N-1)F^{N-2}(r)f(r)\left(v - \hat{b}(r)\right) - F^{N-1}(r) \cdot \hat{b}'(r)\right] \\
&\quad - \left[(N-1)F^{N-2}(r)f(r)\left(r - \hat{b}(r)\right) - F^{N-1}(r) \cdot \hat{b}'(r)\right] \\
&= (N-1)F^{N-2}(r)f(r)(v-r)
\end{aligned}$$

and since $F, f > 0$ for all $r \in (0, \bar{v}]$, $du(r,v)/dr > 0$ when $v > r$ and < 0 when $v < r$, therefore, $u(r,v)$ attains maximum at $r = v$.

Question 4. *In a **first-price, all-pay auction**, the bidders simultaneously submit sealed bids. The highest bid wins the object and every bidder pays the seller the amount of his bid. Consider the independent private values model with symmetric bidders whose values are each distributed according to the distribution function F with density f.*

(a) Find the unique symmetric equilibrium bidding function. Interpret.

ANSWER: Now since a bidder always pays his bid but only gets the object if his bid is the highest, the expected utility by bidding according to $\hat{b}(r)$ when value is v is

$$u(r,v) = F^{N-1}(r)\,v - \hat{b}(r)$$

And the foc is

$$\frac{\partial u(r,v)}{\partial r} = (N-1)\,F^{N-2}(r)\,f(r)\,v - \hat{b}'(r)$$

setting to zero at $r = v$,

$$\begin{aligned}
\hat{b}'(v) &= (N-1)\,F^{N-2}(v)\,f(v)\,v = v\left(F^{N-1}(v)\right)' \\
\int d\hat{b}(v) &= \int_0^v x\,d\left(F^{N-1}(x)\right) + k \\
\hat{b}(v) &= vF^{N-1}(v) - \int_0^v F^{N-1}(x)\,dx + k
\end{aligned}$$

and $\hat{b}(0) = 0$ fixes $k = 0$. Therefore, the unique symmetric bidding function is

$$\hat{b}^{\text{APFPA}}(v) = F^{N-1}(v)\left[v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{N-1} dx\right]$$

Interpretation is in next part.

(b) Do bidders bid higher or lower than in a first-price auction?

ANSWER: We see that $\hat{b}^{\text{APFPA}}(v) = F^{N-1}(v)\,\hat{b}^{\text{FPA}}(v)$, and $F(v) \leq 1$, so bidders bid lower than in a first price auction, and bid by exactly a factor equal to the probability of winning.

(c) Find an expression for the seller's expected revenue.

ANSWER: The seller's expected revenue is the sum of the bids for all agents since everyone pays, i.e.

$$\begin{aligned}
R^{\text{APFPA}} &= N\int_0^{\bar{v}} \hat{b}^{\text{APFPA}}(v)\,dF(v) \\
&= N\int_0^{\bar{v}} F^{N-1}(v)\left[v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{N-1} dx\right] dF(v).
\end{aligned}$$

(d) Both with and without revenue equivalence theorem, show that the seller's expected revenue is the same as in a first-price auction.

ANSWER: Without revenue equivalence theorem, we show directly that $R^{\text{APFPA}} = R^{\text{FPA}}$.

$$R^{\text{FPA}} = N \int_0^{\bar{v}} F^{N-1}(v)\, \hat{b}^{\text{FPA}}(v)\, dF(v) = N \int_0^{\bar{v}} \hat{b}^{\text{APFPA}}(v)\, dF(v)$$

which is exactly R^{APFPA} as expressed above.

Since both FPA and APFPA assign the object to the bidder with the highest value, and are both incentive-compatible with bidder with value zero indifferent between the mechanism as their expected payoff is zero, by RET, the two mechanisms generate the same expected revenues.