Online Appendix for "A Marriage-Market Perspective of the College Gender Gap and the Gender Pay Gap"

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The online appendix contains an enriched model of Zhang (2016), "A Marriage-Market Perspective of the College Gender Gap and the Gender Pay Gap," `http://ssrn.com/abstract=2770854`*. In the enriched model, women do not pay an additional cost for career investment; instead, after their career investment, they delay marriage and may have a lower reproductive fitness which adversely affects their marriage surplus, effective paying an endogenous cost associated with reproductive decline and career investment. Because women are categorized by two dimensions - income and reproductive fitness, the equilibrium of the enriched model is much more complicated to describe than that of the basic model (e.g. 14 possible cases of stable matchings in the enriched model versus three in the basic model), but all the results in the main text hold in the enriched model. An additional theorem about the efficiency of the equilibrium investments is stated and proved. Since the equilibrium is much more complicated to describe in the enriched model, I elect to use the simpler model in the main text and include the enriched model separately in the online appendix as an extension and generalization.*

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I introduce an enriched model building on the basic model. I state the modified model and restate the results (lemmas, theorems, and propositions) in the context of the enriched model, and show additional robustness checks and an additional theorem regarding the social efficiency of the equilibrium investments. For convenient comparison with the basic model, I maintain the same orderings and numberings as in the main text for sections and subsections as well as for equations, lemmas, theorems, and propositions.

I. Model

The basic setting is the same as that in the basic model: there is an infinite number of periods, and unit masses of men and women who are endowed with investment costs c_m and c_w which are distributed according to continuous and strictly increasing distributions F_m and F_w make college and career investment decisions.

A. College and Career Investments

The investment opportunities are the same as those in the basic model: men and women decide whether or not to invest in college in the first period of their lives, and, depending on the outcome of the college investment, decide whether or not to invest in career in the second period of their lives. The only difference is that there is no cost k for women to make a career investment. Instead, the cost associated with women's career investment is endogenously generated from the reproductive difference, which I will describe next.

B. Induced Marriage-Type Distributions

The enriched model only departs from the basic model starting from the organization of the marriage market. While all the investment opportunities are the same for men and women, the lengths they stay reproductively fit are different: any man who enters the marriage market in any of the three periods of his life and any woman who enters the marriage market in the first or second period of her life are reproductively fit for sure, but any woman who enters the marriage market in the third period of her life may have a lower reproductive fitness. Let r represent the probability that a woman who enters the marriage market in the third period of her life is reproductively fit and $1 - r$ the probability that she is less fit.[1]

The marriage surplus s a couple generates depends on the husband's income, the wife's income and her reproductive fitness. Consequently, there are two payoff-relevant types of men and four types of women in the marriage market. Men in the marriage market are distinguished by their income alone: a high-income man is of type H and a low-income man is of type L. Women in the marriage market are distinguished by both their income and reproductive fitness: a high-income reproductively fit woman is of type $\mathcal{H}$, a high-income reproductively less fit woman is of type $\hbar$, a low-income fit woman is of type $\mathcal{L}$, and a low-income less fit woman is of type ℓ.

[1] If reproductive fitness is not observable before marriage so that all age 3 women are identically treated in their reproductive dimension, then the model in which $r = 0$ captures such possibility.

Stationary and symmetric investment strategies σ_m and σ_w *induce* stationary marriage-type masses $G_m = (G_{mH}, G_{mL})$ and $G_w = (G_{w\mathcal{H}}, G_{w\hbar}, G_{w\mathcal{L}}, G_{w\ell})$:

$$G_{mH} = \int_0^{\bar{c}} \sigma_{m1}(c_m)[p_m + (1-p_m)\sigma_{m2}(c_m)p_m]dF_m(c_m).$$

$$G_{mL} = \int_0^{\bar{c}} \{1 - \sigma_{m1}(c_m) \\ + \sigma_{m1}(c_m)(1-p_m)[1 - \sigma_{m2}(c_m) + \sigma_{m2}(c_m)(1-p_m)]\}dF_m(c_m).$$

$$G_{w\mathcal{H}} = \int_0^{\bar{c}} \sigma_{w1}(c_w)[p_w + (1-p_w)\sigma_{w2}(c_w)p_w r]dF_w(c_w).$$

$$G_{w\hbar} = \int_0^{\bar{c}} \sigma_{w1}(c_w)(1-p_w)\sigma_{w2}(c_w)p_w(1-r)dF_w(c_w).$$

$$G_{w\mathcal{L}} = \int_0^{\bar{c}} \{1-\sigma_{w1}(c_w)+\sigma_{w1}(c_1)(1-p_w)[1-\sigma_{w2}(c_w)+\sigma_{w2}(c_w)(1-p_w)r]\}dF_w(c_w).$$

$$G_{w\ell} = \int_0^{\bar{c}} \sigma_{w1}(c_1)(1-p_w)\sigma_{w2}(c_w)(1-p_w)(1-r)dF_w(c_w).$$

C. The Marriage Market

MARRIAGE SURPLUS

The marriage surplus is represented by the eight different combinations of marriage-types: $s_{H\mathcal{H}}$, $s_{H\mathcal{L}}$, $s_{H\hbar}$, $s_{H\ell}$, $s_{L\mathcal{H}}$, $s_{L\mathcal{L}}$, $s_{L\hbar}$, and $s_{L\ell}$. Throughout the paper, assume that the surplus is non-negative, that singles generate no marriage surplus, and that higher income and higher reproductive fitness generate strictly higher surplus (namely, $s_{H\tau_w} > s_{L\tau_w}$, and $s_{\tau_m\mathcal{H}} > s_{\tau_m\mathcal{L}}$, $s_{\tau_m\hbar} > s_{\tau_m\ell}$, where $\tau_m \in \{H, L\}$ and $\tau_w \in \{\mathcal{H}, \mathcal{L}, \hbar, \ell\}$). I do not restrict the order of $s_{\tau_m\mathcal{L}}$ and $s_{\tau_m\hbar}$. Furthermore, I make the following supermodularity assumptions on the marriage surplus.

ASSUMPTION 1 (Income-Income Supermodularity): *A couple's marriage surplus is strictly supermodular in the husband's income and the wife's income:* $s_{H\mathcal{H}} - s_{H\mathcal{L}} > s_{L\mathcal{H}} - s_{L\mathcal{L}}$ *and* $s_{H\hbar} - s_{H\ell} > s_{L\hbar} - s_{L\ell}$.

ASSUMPTION 2 (Income-Fitness Supermodularity): *A couple's marriage surplus is strictly supermodular in the husband's income and the wife's reproductive fitness:* $s_{H\mathcal{H}} - s_{H\hbar} > s_{L\mathcal{H}} - s_{L\hbar}$ *and* $s_{H\mathcal{L}} - s_{H\ell} > s_{L\mathcal{L}} - s_{L\ell}$.

These supermodularity assumptions will help to pin down the stable matching patterns, and their implications on the main result (Proposition 2) will be discussed in Section IV.

Let me demonstrate an income-income and income-fitness supermodular surplus function that is consistent with an underlying household utility maximization problem using the simplest possible household problem and utility functional form. The household utility maximization problem also justifies the transferable utility assumption in the marriage market. Consider a man with income y_m and a woman with income y_w and reproductive fitness R. The couple chooses the allocation of the man's private good q_m, the woman's private good q_w, and the couple's public good Q to maximize the sum of their utilities $Rq_mQ + Rq_wQ$ subject to the budget constraint $q_m + q_w + Q = y_m + y_w$. The utility function in the presence of a public good in the household enables transferable utilities through the allocation of private goods. The maximized surplus of the couple given their incomes y_m and y_w is calculated as $R(y_m + y_w)^2/4$ (the algebra is as follows: first, plug $q_m + q_w = y_m + y_w - Q$ into the sum of the utilities to get $R(y_m + y_w - Q)Q$; then, derive the optimal Q^* that maximizes the total utility: via first order condition, $Q^*(y_m, y_w, R) = (y_m + y_w)/2$; finally, plugging the optimal solution $q^*_m(y_m, y_w, R) + q^*_w(y_m, y_w, R) = Q^*(y_m, y_w, R) = (y_m + y_w)/2$ back in to derive the indirect surplus function, $s(y_m, y_w, R) = R[(y_m + y_w)/2]^2 = R(y_m + y_w)^2/4$). The derived surplus function is increasing in y_m, y_w and R, and is income-income supermodular as well as income-fitness supermodular.

Stable Marriage Market Outcome: Matchings and Payoffs

In a marriage market described by (G_m, G_w), men and women match and bargain over their marriage surplus. A *stable outcome* of the marriage market (G_m, G_w) consists of a *matching* described by a measure G on the matching-types $\{H, L\} \times \{\mathcal{H}, \mathcal{L}, \hbar, \ell\}$ with marginals G_m and G_w, and marriage payoffs $v_m = (v_{mH}, v_{mL})$ and $v_w = (v_{w\mathcal{H}}, v_{w\mathcal{L}}, v_{w\hbar}, v_{w\ell})$.[2] $G_{\tau_m\tau_w}$ describes the measure of couples with a type τ_m husband and a type τ_w wife. Stable marriage payoffs satisfy the following stability conditions.

1) (Individual rationality). Every agent receives weakly more than being single: $v_{m\tau_m} \geq 0$ and $v_{w\tau_w} \geq 0$.

2) (Pairwise efficiency). Every matched couple divides the entire surplus: $v_{m\tau_m} + v_{w\tau_w} = s_{\tau_m\tau_w}$ if $G_{\tau_m\tau_w} > 0$.

3) (No blocking pair). No man and no woman who are not married to each other prefer to marry each other because there is no division of surplus that would make both of them strictly better off: $v_{m\tau_m} + v_{w\tau_w} \geq s_{\tau_m\tau_w}$.

[2]Some marriage-types may not be in the support of the induced marriage-type distributions. For example, when no man goes to college, there is zero mass of high-income men in the marriage market, or when no woman makes a career investment, there is zero mass of high-income less-fit and low-income less-fit women. In such situations, stability conditions do not restrict the out-of-support marriage-type's marriage payoff. I define for any out-of-support $\tau_m \in \{H, L\}$, $v_{m\tau_m} = \max_{\tau_w \in \text{supp}(G_w)}(s_{\tau_m\tau_w} - v_{w\tau_w})$, and for any out-of-support τ_w, $v_{w\tau_w} = \max_{\tau_m \in \text{supp}(G_m)}(s_{\tau_m\tau_w} - v_{m\tau_m})$. Furthermore, I assume that agents have rational expectations and know the stable marriage payoffs if they enter the marriage market as out-of-support marriage-types. The assumption is purely for the sake of completeness. When everyone maximizes utility, the situations with missing marriage-types will not arise.

For any marriage market (G_m, G_w), a stable outcome (G, v_m, v_w) exists, by Theorem 2 in Gretsky, Ostroy and Zame (1992).

I do not specify each individual's partner in the stable matching, and instead I only use a measure G to describe the total mass of couples of different combinations of marriage-types, represented by eight numbers: $G_{H\mathcal{H}}$, $G_{H\mathcal{L}}$, $G_{H\hbar}$, $G_{H\ell}$, $G_{L\mathcal{H}}$, $G_{L\mathcal{L}}$, $G_{L\hbar}$, and $G_{L\ell}$. Since all agents in the model are anonymous, I say two matchings are equivalent if the eight numbers are the same. I do not need to define the exact pairwise matching in the marriage market with continuum of agents because people make investment decisions only based on their (expected) stable marriage payoffs.

D. *Summary*

The *primitives* of the model are: investment cost distributions (F_m, F_w), success probabilities (p_m, p_w), incomes ($y_{m\mathrm{H}}$, $y_{m\mathrm{L}}$, $y_{w\mathrm{H}}$, $y_{w\mathrm{L}}$), and marriage surplus ($s_{H\mathcal{H}}$, $s_{H\mathcal{L}}$, $s_{H\hbar}$, $s_{H\ell}$, $s_{L\mathcal{H}}$, $s_{L\mathcal{L}}$, $s_{L\hbar}$, $s_{L\ell}$).

II. Equilibrium

The definition of the equilibrium is the same as in the basic model.

A. *Optimal Investments*

LEMMA 1: *Given marriage payoffs v_m and v_w, let c^*_m, c^*_{w1}, and c^*_{w2} be defined by equations*

(1) $$c^*_m \equiv p_m(y_{m\mathrm{H}} - y_{m\mathrm{L}}) + p_m(v_{mH} - v_{mL}),$$

(2) $$c^*_{w1} \equiv p_w(y_{w\mathrm{H}} - y_{w\mathrm{L}}) + p_w(v_{w\mathcal{H}} - v_{w\mathcal{L}}),$$

and

(3)
$$c^*_{w2} \equiv p_w(y_{w\mathrm{H}} - y_{w\mathrm{L}}) + rp_w(v_{w\mathcal{H}} - v_{w\mathcal{L}}) + (1-r)(p_w(v_{w\hbar} - v_{w\mathcal{L}}) + (1-p_w)(v_{w\ell} - v_{w\mathcal{L}})),$$

respectively. Cost $c_m \le c^*_m$ men invest in both college and career, and cost $c_m > c^*_m$ men do not invest at all. Cost $c_w \le c^*_{w2}$ women make both college and career investments, cost $c_w \in (c^*_{w2}, c^*_{w1}]$ women make college investments but do not make career investments, and cost $c_w > c^*_{w1}$ women do not invest at all.

It is worth noting that, the fact that optimal investment strategies can be characterized by cutoff costs is independent of the supermodularity assumptions on marriage surplus. Surplus monotonicity is sufficient. Agents make their decisions based on their payoffs only and not on their partners, as their own marriage-type determines their bargaining power in the marriage market.

The relevant payoff differences that determine the optimal cutoffs are $v_{mH} - v_{mL}$, $v_{w\mathcal{H}} - v_{w\mathcal{L}}$, $v_{w\hbar} - v_{w\mathcal{L}}$, and $v_{w\ell} - v_{w\mathcal{L}}$. These payoff differences depend on the marriage-type distributions in the marriage market, which I will characterize next.

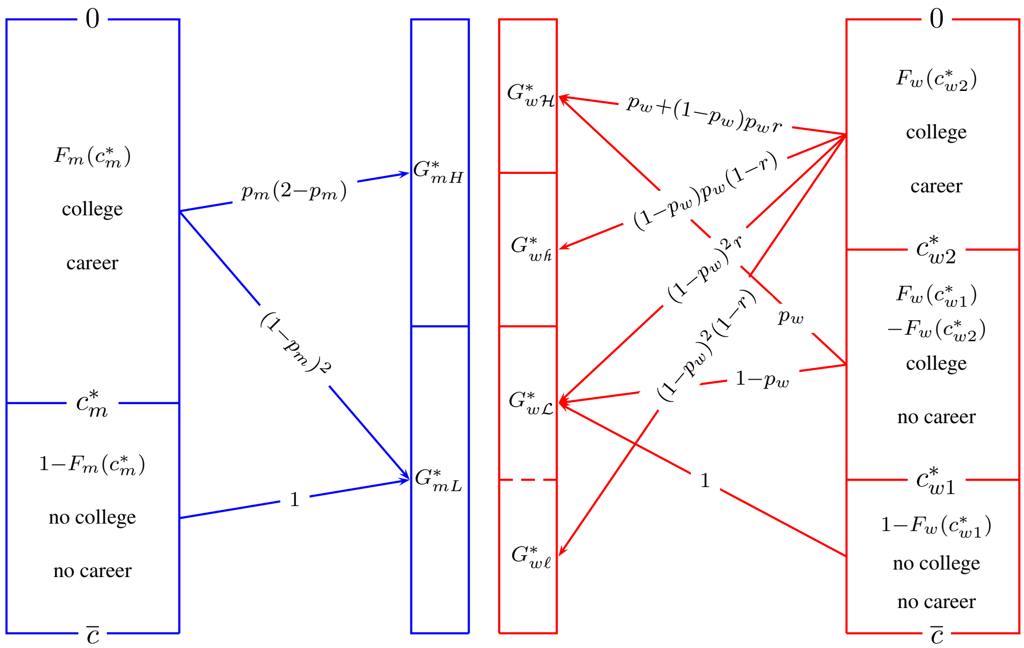

FIGURE 1. OPTIMAL INVESTMENTS AND INDUCED MARRIAGE-TYPE DISTRIBUTIONS.

Note: Cost $c_m < c_m^*$ men invest in college and career; fraction $(1-p_m)^2$ of them become type L and fraction $p_m(2-p_m)$ become type H. Cost $c_m \geq c_m^*$ men do not invest in college or in career and all become type L. Cost $c_w < c_{w2}^*$ women invest both in college and in career; fraction $p_w + (1-p_w)p_w r$ of them become type H, fraction $(1-p_w)p_w(1-r)$ become type h, fraction $(1-p_w)^2 r$ become type L, and fraction $(1-p_w)^2 r$ become type l. Cost $c_{w2}^* \leq c_w < c_{w1}^*$ women invest in college but not in career; fraction p_w become type H and fraction $1-p_w$ become type L. Cost $c_w \geq c_{w1}^*$ women do not invest and all become type L.

B. *Induced Marriage-Type Distributions*

When agents play the optimal investment strategies characterized by cutoff costs c_m^*, c_{w1}^*, and c_{w2}^*, the induced marriage-type distributions are more easily characterized than the distributions induced by arbitrary investments. Figure 1 illustrates optimal investments characterized by the cutoff costs and the marriage-type distributions induced by the optimal investments.

LEMMA 2: *When agents play investment strategies characterized by cutoff costs c_m^*, c_{w1}^*, and c_{w2}^*, the induced income distributions are characterized by*

$$G^*_{mH} = F_m(c_m^*)p_m(2-p_m). \tag{4}$$

$$G^*_{mL} = 1 - G^*_{mH} = 1 - F_m(c_m^*)p_m(2-p_m), \tag{5}$$

$$G^*_{w\mathcal{H}} = F_w(c_{w1}^*)p_w + F_w(c_{w2}^*)(1-p_w)p_w r, \tag{6}$$

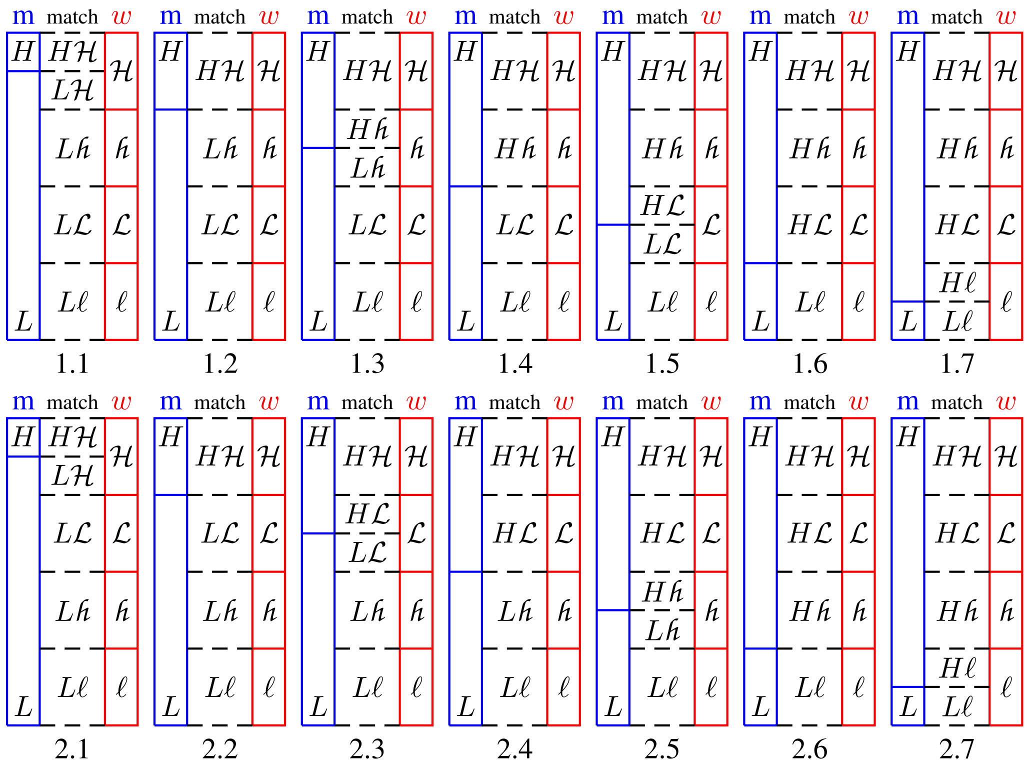

FIGURE 2. STABLE MATCHINGS UNDER INCOME-INCOME SUPERMODULARITY AND INCOME-FITNESS SUPERMODULARITY.

Note: Type $\mathcal{H}$ women almost always marry higher-income husbands than other women do. Type ℓ women almost always marry lower-income husbands than other women do. When $s_{H\hbar} + s_{L\mathcal{L}} > s_{H\mathcal{L}} + s_{L\hbar}$, a type $\hbar$ woman almost always marries a weakly higher-income husband than a type $\mathcal{L}$ woman does (Markets 1.1-1.7). When $s_{H\hbar}+s_{L\mathcal{L}} < s_{H\mathcal{L}}+s_{L\hbar}$, a type $\hbar$ woman almost always marries a weakly lower-income husband than a type $\mathcal{L}$ woman does (Markets 2.1-2.7).

$$G^*_{w\hbar} = F_w(c^*_{w2})(1 - p_w)p_w(1 - r), \tag{7}$$

$$G^*_{w\mathcal{L}} = 1 - F_w(c^*_{w1}) + [F_w(c^*_{w1}) - F_w(c^*_{w2})](1 - p_w) + F_w(c^*_{w2})(1 - p_w)^2 r, \tag{8}$$

$$G^*_{w\ell} = (1 - p_w)^2(1 - r)F_w(c^*_{w2}). \tag{9}$$

C. *Stable Matchings*

By Theorem 2 in Gretsky, Ostroy and Zame (1992), stable matching exists given the induced marriage-type distributions and maximizes the population's total marriage surplus, but to further characterize the stable matching pattern of who marries whom, we need to impose additional restrictions on the marriage surplus, in particular the supermodularity Assumptions 1 and 2.

LEMMA 3: *The following conditions are satisfied under any stable matching.*

(a). Suppose that Assumption 1 income-income supermodularity holds. A type $\mathcal{H}$ woman almost always marries a higher-income husband than a type $\mathcal{L}$ woman does, and a type $\hbar$ woman almost always marries a higher-income husband than a type ℓ woman does.

(b). Suppose that Assumption 2 income-fitness supermodularity holds. A type $\mathcal{H}$ woman almost always marries a higher-income husband than a type $\hbar$ woman does, and a type $\mathcal{L}$ woman almost always marries a higher-income husband than a type ℓ woman does.

(c). If $s_{H\hbar} + s_{L\mathcal{L}} > s_{H\mathcal{L}} + s_{L\hbar}$, a type $\hbar$ woman almost always marries a higher-income husband than a type $\mathcal{L}$ woman does. If $s_{H\hbar} + s_{L\mathcal{L}} < s_{H\mathcal{L}} + s_{L\hbar}$, a type $\hbar$ woman almost always marries a lower-income husband than a type L woman does.

PROOF:

All three lemmas take the following form: for $\tau_w, \tau'_w \in \{\mathcal{H}, \mathcal{L}, \hbar, \ell\}$, when $s_{H\tau_w} + s_{L\tau'_w} > s_{H\tau'_w} + s_{L\tau_w}$, a type τ_w woman almost always marries a higher-income husband than a type τ'_w woman does. In Lemma 3a, $\tau_w = \mathcal{H}, \tau'_w = \mathcal{L}$, or $\tau_w = \hbar, \tau'_w = \ell$. In Lemma 3b, $\tau_w = \mathcal{H}, \tau'_w = \hbar$, or $\tau_w = \mathcal{L}, \tau'_w = \ell$. In Lemma 3c, $\tau_w = \hbar, \tau'_w = \mathcal{L}$, or $\tau_w = \mathcal{L}, \tau'_w = \hbar$.

It suffices to show that for $\tau_w, \tau'_w \in \{\mathcal{H}, \mathcal{L}, \hbar, \ell\}$, when $s_{H\tau_w} + s_{L\tau'_w} > s_{H\tau'_w} + s_{L\tau_w}$, there is a zero mass of either (H, τ'_w) or (L, τ_w) couples. Suppose by way of contradiction positive masses of both (H, τ'_w) and (L, τ_w) couples. By pairwise efficiency, $v_{mH} + v_{w\tau'_w} = s_{H\tau'_w}$ and $v_{mL} + v_{w\tau_w} = s_{L\tau_w}$. In addition, by the no blocking pair condition, $v_{mH} + v_{w\tau_w} \geq s_{H\tau_w}$ and $v_{mL} + v_{w\tau'_w} \geq s_{L\tau'_w}$. The equalities and inequalities together yield $s_{H\tau'_w} + s_{L\tau_w} \geq s_{H\tau_w} + s_{L\tau'_w}$, which contradicts the assumption $s_{H\tau'_w} + s_{L\tau_w} < s_{H\tau_w} + s_{L\tau'_w}$.
QED.

Lemmas 3a and 3b follow directly from the standard results of supermodularity and positive-assortative matching since Becker (1973). Combining these two lemmas, a type $\mathcal{H}$ woman almost always marries a weakly higher-income man than a type $\mathcal{L}$, $\hbar$, or ℓ woman does, and that a type ℓ woman almost always marries a weakly lower-income man than a $\mathcal{H}$, $\mathcal{L}$, or $\hbar$ woman. Type $\mathcal{H}$ women are "ranked" the highest in terms of their marriage prospects measured by spousal income, and type ℓ women are "ranked" the lowest. However, the ranking of all women is indeterminate without further assumption on the marriage surplus about whether a type $\hbar$ woman marries a higher-income or a lower-income man than a type $\mathcal{L}$. The condition that distinguishes the two possibilities is whether $(s_{H\hbar} + s_{L\mathcal{L}}) - (s_{H\mathcal{L}} + s_{L\hbar})$ is positive or negative.

Depending on the relative abundance of high-income men, there are 14 different stable matchings, shown in Figure 2. For example (Market 1.5), when $s_{H\hbar} + s_{L\mathcal{L}} > s_{H\mathcal{L}} + s_{L\hbar}$ and $G_{w\mathcal{H}} + G_{w\hbar} < G_{mH} < G_{w\mathcal{H}} + G_{w\hbar} + G_{w\mathcal{L}}$, type $\mathcal{H}$, $\hbar$, $\mathcal{L}$, ℓ women are ordered in likelihood to match with type H men, and there are positive masses of $(H, \mathcal{H})$, $(H, \hbar)$

and $(H, \mathcal{L})$ couples and $(L, \mathcal{L})$ and (L, ℓ) couples. The marriage markets with different stable matchings also have different stable marriage payoffs, characterized next.

D. *Stable Marriage Payoffs*

LEMMA 4: *In the marriage market, stable marriage payoff differences ($v_{mH} - v_{mL}$, $v_{w\mathcal{H}} - v_{w\mathcal{L}}$, $v_{w\hbar} - v_{w\mathcal{L}}$, $v_{w\hbar} - v_{w\ell}$) are specified case by case in Table 2.*

The stable marriage payoffs are derived as follows. Under all possible stable matchings, type L men and type ℓ women are matched. Therefore, $v_{mL} + v_{w\ell} = s_{L\ell}$. However, when $s_{L\mathcal{L}}$ is positive, the division of surplus between type L men and type ℓ women is indeterminate. Both v_{mL} and $v_{w\ell}$ can take on value between 0 and $s_{L\ell}$ as long as the two payoffs add up to $s_{L\ell}$. Consequently, the stable marriage payoffs cannot be uniquely pinned down. Nonetheless, they can be characterized by v_{mL} and $v_{w\mathcal{L}}$, and four stable marriage payoff differences $v_{mH} - v_{mL}$, $v_{w\mathcal{H}} - v_{w\mathcal{L}}$, $v_{w\hbar} - v_{w\mathcal{L}}$, and $v_{w\ell} - v_{w\mathcal{L}}$.

These four marriage payoff differences can be derived from the two stability conditions - pairwise efficiency and no blocking pair. Pairwise efficiency states that any married couple divides its surplus: if there is a positive mass of (τ_m, τ_w) couples in the stable matching, then $v_{m\tau_m} + v_{w\tau_w} = s_{\tau_m\tau_w}$. The no blocking pair condition states that if a man and a woman are not married to each other, they get jointly more in their current marriages than the hypothetical surplus they would get by marrying each other: if there is no positive mass of (τ_m, τ_w) couples in the stable matching, then $v_{m\tau_m} + v_{w\tau_w} \geq s_{\tau_m\tau_w}$. There are $2 \times 4 = 8$ possible couples by marriage-types. Eight equalities and inequalities can be easily derived from Figure 2 with the couples appearing in the figure assigned an equality and the couples not appearing in the figure being assigned a weak inequality; they are summarized in Table 1. These equalities and inequalities conditions pin down the four marriage payoff differences, as follows. The results are summarized in Table 2.

There are two scenarios under which $v_{mH} - v_{mL}$ is derived. First, there exists a woman's marriage type $\tau_w \in \{\mathcal{H}, \mathcal{L}, \hbar, \ell\}$ such that there is a positive mass of (H,τ_w) couples and there is a positive mass of (L, τ_w) couples. Then, $v_{mH} + v_{w\tau_w} = s_{H\tau_w}$ and $v_{mL} + v_{w\tau_w} = s_{L\tau_w}$ imply

$$v_{mH} - v_{mL} = (s_{H\tau_w} - v_{w\tau_w}) - (s_{L\tau_w} - v_{w\tau_w}) = s_{H\tau_w} - s_{L\tau_w}.$$

Second, there does not exist such a woman's type that positive masses of both type H and L men marry this type of woman. Rank women's types as $\{\mathcal{H}, \hbar, \mathcal{L}, \ell\}$ in Markets 1.1-1.7 and as $\{\mathcal{H}, \mathcal{L}, \hbar, \ell\}$ in Markets 2.1-2.7. Let τ_w be the lowest type such that there's a positive mass of (H, τ_w) couples. Then we have the equality $v_{mH} + v_{w\tau_w} = s_{H\tau_w}$ and inequality $v_{mL} + v_{w\tau_w} \geq s_{L\tau_w}$ from the two stability conditions, which together imply

$$v_{mH} - v_{mL} \leq (s_{H\tau_w} - v_{w\tau_w}) - (s_{L\tau_w} - v_{w\tau_w}) = s_{H\tau_w} - s_{L\tau_w}.$$

Let τ_w' be the highest type such that there is a positive mass of (L, τ_w') couples. Then we have the equality $v_{mL} + v_{w\tau_w'} = s_{L\tau_w'}$ and inequality $v_{mH} + v_{w\tau_w'} \geq s_{H\tau_w'}$ from the two

TABLE 1—STABILITY CONDITIONS IN MARKETS 1.1 TO 1.7 AND IN MARKETS 2.1 TO 2.7.

	1.1	1.2	1.3	1.4	1.5	1.6	1.7	
$v_{mH} + v_{w\mathcal{H}}$	$=$	$=$	$=$	$=$	$=$	$=$	$=$	$s_{H\mathcal{H}}$
$v_{mH} + v_{w\hbar}$	$\geq$	$\geq$	$=$	$=$	$=$	$=$	$=$	$s_{H\hbar}$
$v_{mH} + v_{w\mathcal{L}}$	$\geq$	$\geq$	$\geq$	$\geq$	$=$	$=$	$=$	$s_{H\mathcal{L}}$
$v_{mH} + v_{w\ell}$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$=$	$s_{H\ell}$
$v_{mL} + v_{w\mathcal{H}}$	$=$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$s_{L\mathcal{H}}$
$v_{mL} + v_{w\hbar}$	$=$	$=$	$=$	$\geq$	$\geq$	$\geq$	$\geq$	$s_{L\hbar}$
$v_{mL} + v_{w\mathcal{L}}$	$=$	$=$	$=$	$=$	$=$	$\geq$	$\geq$	$s_{L\mathcal{L}}$
$v_{mL} + v_{w\ell}$	$=$	$=$	$=$	$=$	$=$	$=$	$=$	$s_{L\ell}$
	2.1	2.2	2.3	2.4	2.5	2.6	2.7	
$v_{mH} + v_{w\mathcal{H}}$	$=$	$=$	$=$	$=$	$=$	$=$	$=$	$s_{H\mathcal{H}}$
$v_{mH} + v_{w\mathcal{L}}$	$\geq$	$\geq$	$=$	$=$	$=$	$=$	$=$	$s_{H\mathcal{L}}$
$v_{mH} + v_{w\hbar}$	$\geq$	$\geq$	$\geq$	$\geq$	$=$	$=$	$=$	$s_{H\hbar}$
$v_{mH} + v_{w\ell}$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$=$	$s_{H\ell}$
$v_{mL} + v_{w\mathcal{H}}$	$=$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$s_{L\mathcal{H}}$
$v_{mL} + v_{w\mathcal{L}}$	$=$	$=$	$=$	$\geq$	$\geq$	$\geq$	$\geq$	$s_{L\mathcal{L}}$
$v_{mL} + v_{w\hbar}$	$=$	$=$	$=$	$=$	$=$	$\geq$	$\geq$	$s_{L\hbar}$
$v_{mL} + v_{w\ell}$	$=$	$=$	$=$	$=$	$=$	$=$	$=$	$s_{L\ell}$

stability conditions, which together imply

$$v_{mH} - v_{mL} \geq (s_{H\tau'_w} - v_{w\tau'_w}) - (s_{L\tau'_w} - v_{w\tau'_w}) = s_{H\tau'_w} - s_{L\tau'_w}.$$

Altogether, the two inequalities imply

$$s_{H\tau'_w} - s_{L\tau'_w} \leq v_{mH} - v_{mL} \leq s_{H\tau_w} - s_{L\tau_w},$$

so for any $\lambda \in [0, 1]$,

$$v_{mH} - v_{mL} = (1 - \lambda)(s_{H\tau_w} - s_{L\tau_w}) + \lambda(s_{H\tau'_w} - s_{L\tau'_w}).$$

There are three possibilities for calculating $v_{w\tau_w} - v_{w\mathcal{L}}$. First, if there is a type $\tau_m \in \{H, L\}$ such that there is a positive mass of (τ_m, τ_w) couples and there is a positive mass of $(\tau_m, \mathcal{L})$ couples, then $v_{m\tau_m} + v_{w\tau_w} = s_{\tau_m\tau_w}$ and $v_{m\tau_m} + v_{w\mathcal{L}} = s_{\tau_m\mathcal{L}}$, which together imply

$$v_{w\tau_w} - v_{w\mathcal{L}} = (s_{\tau_m\tau_w} - v_{m\tau_m}) - (s_{\tau_m\mathcal{L}} - v_{m\tau_m}) = s_{\tau_m\tau_w} - s_{\tau_m\mathcal{L}}.$$

Second, if there is no such type that $\tau_m \in \{H, L\}$ that there is a positive mass of (τ_m, τ_w) couples and there is a positive mass of $(\tau_m, \mathcal{L})$ couples, then almost all τ_w women marry τ_m men and almost all $\mathcal{L}$ women marry $\tau'_m \neq \tau_m$ men. This scenario can be divided into two possibilities. One, there exists a type $\tau'_w \in \{\mathcal{H}, \mathcal{L}, \hbar, \ell\}$ such that there is a positive mass of (τ_m, τ'_w) and there's a positive mass of (τ'_m, τ'_w) couples. Since there are positive mass of (τ_m, τ_w) and (τ_m, τ'_w), couples,

$$v_{w\tau_w} - v_{w\tau'_w} = (s_{\tau_m\tau_w} - v_{m\tau_m}) - (s_{\tau_m\tau'_w} - v_{m\tau_m}) = s_{\tau_m\tau_w} - s_{\tau_m\tau'_w}.$$

Since there are positive masses of (τ'_m, τ'_w) and $(\tau'_m, \mathcal{L})$ couples,

$$v_{w\tau'_w} - v_{w\mathcal{L}} = (s_{\tau'_m\tau'_w} - v_{m\tau'_m}) - (s_{\tau'_m\mathcal{L}} - v_{m\tau'_m}) = s_{\tau'_m\tau'_w} - s_{\tau'_m\mathcal{L}}.$$

Combining the two equations yields,

$$v_{w\tau_w} - v_{w\mathcal{L}} = s_{\tau_m\tau_w} - s_{\tau_m\tau'_w} + s_{\tau'_m\tau'_w} - s_{\tau'_m\mathcal{L}}.$$

Two, there does not exist a type $\tau_m \in \{H, L\}$ such that there is a positive mass of (τ_m, τ_w) couples and there's a positive mass of $(\tau_m, \mathcal{L})$ couples and there does not exist a type $\tau'_w \in \{\mathcal{H}, \mathcal{L}, \hbar, \ell\}$ such that there is a positive mass of (τ'_m, τ'_w) couples. Almost all τ_w women marry type τ_m men and almost all $\mathcal{L}$ women marry type $\tau'_m \neq \tau_m$ men. Hence, $v_{m\tau_m} + v_{w\tau_w} = s_{\tau_m\tau_w}$ and $v_{m\tau_m} + v_{w\mathcal{L}} \geq s_{\tau_m\mathcal{L}}$ imply

$$v_{w\tau_w} - v_{w\mathcal{L}} \leq (s_{\tau_m\tau_w} - v_{m\tau_m}) - (s_{\tau_m\mathcal{L}} - v_{m\tau_m}) = s_{\tau_m\tau_w} - s_{\tau_m\mathcal{L}}.$$

$v_{m\tau'_m} + v_{w\tau_w} \geq s_{\tau'_m\tau_w}$ and $v_{m\tau'_m} + v_{w\mathcal{L}} = s_{\tau'_m\mathcal{L}}$ imply

$$v_{w\tau_w} - v_{w\mathcal{L}} \geq (s_{\tau'_m\tau_w} - v_{m\tau'_m}) - (s_{\tau'_m\mathcal{L}} - v_{m\tau'_m}) = s_{\tau'_m\tau_w} - s_{\tau'_m\mathcal{L}}.$$

Together,

$$s_{\tau'_m\tau_w} - s_{\tau'_m\mathcal{L}} \leq v_{w\tau_w} - v_{w\mathcal{L}} \leq s_{\tau_m\tau_w} - s_{\tau_m\mathcal{L}}.$$

The stable marriage payoff differences in each Market derived according to the process described above are summarized in Table 2. I illustrate the derivations by two examples, Market 1.5 and Market 1.4. In Market 1.5, since there is a positive mass of $(H, \mathcal{H})$, $(H, \hbar)$, $(H, \mathcal{L})$, $(L, \mathcal{L})$ and (L, ℓ) couples, equalities are established for these couples.

Since both H and L men marry $\mathcal{L}$ women, $v_{mH} + v_{w\mathcal{L}} = s_{H\mathcal{L}}$ and $v_{mL} + v_{w\mathcal{L}} = s_{L\mathcal{L}}$ imply

$$v_{mH} - v_{mL} = (s_{H\mathcal{L}} - v_{w\mathcal{L}}) - (s_{L\mathcal{L}} - v_{w\mathcal{L}}) = s_{H\mathcal{L}} - s_{L\mathcal{L}}.$$

Since both $\mathcal{H}$ and $\mathcal{L}$ women marry H men, $v_{mH} + v_{w\mathcal{H}} = s_{H\mathcal{H}}$ and $_{mH} + v_{w\mathcal{L}} = s_{H\mathcal{L}}$ imply

$$v_{w\mathcal{H}} - v_{w\mathcal{L}} = (s_{H\mathcal{H}} - v_{mH}) - (s_{H\mathcal{L}} - v_{mH}) = s_{H\mathcal{H}} - s_{H\mathcal{L}}.$$

Since $\hbar$ and $\mathcal{L}$ women marry H men, $v_{mH} + v_{w\hbar} = s_{H\hbar}$ and $v_{mH} + v_{w\mathcal{L}} = s_{H\mathcal{L}}$ imply

$$v_{w\hbar} - v_{w\mathcal{L}} = (s_{H\hbar} - v_{mH}) - (s_{H\mathcal{L}} - v_{mH}) = s_{H\hbar} - s_{H\mathcal{L}}.$$

Finally, since both ℓ and $\mathcal{L}$ women marry L men, $v_{mL} + v_{w\ell} = s_{L\ell}$ and $v_{mL} + v_{w\mathcal{L}} = s_{L\mathcal{L}}$

TABLE 2—STABLE MARRIAGE PAYOFF DIFFERENCES UNDER INCOME-INCOME SUPERMODULARITY AND INCOME-FITNESS SUPERMODULARITY.

	$v_{mH} - v_{mL}$	$v_{w\mathcal{H}} - v_{w\mathcal{L}}$	$v_{w\hbar} - v_{w\mathcal{L}}$	$v_{w\ell} - v_{w\mathcal{L}}$
1.1	$s_{H\mathcal{H}} - s_{L\mathcal{H}}$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$	$s_{L\hbar} - s_{L\mathcal{L}}$	$s_{L\ell} - s_{L\mathcal{L}}$
1.2	$(1-\lambda)(s_{H\mathcal{H}} - s_{L\mathcal{H}}) + \lambda(s_{H\hbar} - s_{L\hbar})$	$(1-\lambda)(s_{L\mathcal{H}} - s_{L\mathcal{L}}) + \lambda(s_{H\mathcal{H}} - s_{H\hbar} + s_{L\hbar} - s_{L\mathcal{L}})$	$s_{L\hbar} - s_{L\mathcal{L}}$	$s_{L\ell} - s_{L\mathcal{L}}$
1.3	$s_{H\hbar} - s_{L\hbar}$	$s_{H\mathcal{H}} - s_{H\hbar} + s_{L\hbar} - s_{L\mathcal{L}}$	$s_{L\hbar} - s_{L\mathcal{L}}$	$s_{L\ell} - s_{L\mathcal{L}}$
1.4	$(1-\lambda)(s_{H\hbar} - s_{L\hbar}) + \lambda(s_{H\mathcal{L}} - s_{L\mathcal{L}})$	$(1-\lambda)(s_{H\mathcal{H}} - s_{H\hbar} + s_{L\hbar} - s_{L\mathcal{L}}) + \lambda(s_{H\mathcal{H}} - s_{H\mathcal{L}})$	$(1-\lambda)(s_{L\hbar} - s_{L\mathcal{L}}) + \lambda(s_{H\hbar} - s_{H\mathcal{L}})$	$s_{L\ell} - s_{L\mathcal{L}}$
1.5	$s_{H\mathcal{L}} - s_{L\mathcal{L}}$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$	$s_{H\hbar} - s_{H\mathcal{L}}$	$s_{L\ell} - s_{L\mathcal{L}}$
1.6	$(1-\lambda)(s_{H\mathcal{L}} - s_{L\mathcal{L}}) + \lambda(s_{H\ell} - s_{L\ell})$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$	$s_{H\hbar} - s_{H\mathcal{L}}$	$(1-\lambda)(s_{L\ell} - s_{L\mathcal{L}}) + \lambda(s_{H\ell} - s_{H\mathcal{L}})$
1.7	$s_{H\ell} - s_{L\ell}$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$	$s_{H\hbar} - s_{H\mathcal{L}}$	$s_{H\ell} - s_{H\mathcal{L}}$
2.1	$s_{H\mathcal{H}} - s_{L\mathcal{H}}$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$	$s_{L\hbar} - s_{L\mathcal{L}}$	$s_{L\ell} - s_{L\mathcal{L}}$
2.2	$(1-\lambda)(s_{H\mathcal{H}} - s_{L\mathcal{H}}) + \lambda(s_{H\mathcal{L}} - s_{L\mathcal{L}})$	$(1-\lambda)(s_{L\mathcal{H}} - s_{L\mathcal{L}}) + \lambda(s_{H\mathcal{H}} - s_{H\mathcal{L}})$	$s_{L\hbar} - s_{L\mathcal{L}}$	$s_{L\ell} - s_{L\mathcal{L}}$
2.3	$s_{H\mathcal{L}} - s_{L\mathcal{L}}$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$	$s_{L\hbar} - s_{L\mathcal{L}}$	$s_{L\ell} - s_{L\mathcal{L}}$
2.4	$(1-\lambda)(s_{H\mathcal{L}} - s_{L\mathcal{L}}) + \lambda(s_{H\hbar} - s_{L\hbar})$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$	$(1-\lambda)(s_{L\hbar} - s_{L\mathcal{L}}) + \lambda(s_{H\hbar} - s_{H\mathcal{L}})$	$(1-\lambda)(s_{L\ell} - s_{L\mathcal{L}}) + \lambda(s_{L\ell} - s_{L\hbar} + s_{H\hbar} - s_{H\mathcal{L}})$
2.5	$s_{H\hbar} - s_{L\hbar}$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$	$s_{H\hbar} - s_{H\mathcal{L}}$	$s_{L\ell} - s_{L\hbar} + s_{H\hbar} - s_{H\mathcal{L}}$
2.6	$(1-\lambda)(s_{H\hbar} - s_{L\hbar}) + \lambda(s_{H\ell} - s_{L\ell})$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$	$s_{H\hbar} - s_{H\mathcal{L}}$	$(1-\lambda)(s_{L\ell} - s_{L\hbar} + s_{H\hbar} - s_{H\mathcal{L}}) + \lambda(s_{H\ell} - s_{H\mathcal{L}})$
2.7	$s_{H\ell} - s_{L\ell}$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$	$s_{H\hbar} - s_{H\mathcal{L}}$	$s_{H\ell} - s_{H\mathcal{L}}$

imply

$$v_{w\ell} - v_{w\mathcal{L}} = (s_{L\ell} - v_{mL}) - (s_{L\mathcal{L}} - v_{mL}) = s_{L\ell} - s_{L\mathcal{L}}.$$

The marriage payoff differences in Market 1.5 have been derived from the five equalities based on pairwise efficiency alone (scenarios 1 for both men and women described above). The stable marriage payoff differences in all odd-numbered Markets (1.1, 1.3, etc.) can be similarly derived and uniquely pinned down. However, in even-numbered Markets (1.2, 1.4, etc.), only four types of couples appear in the stable matchings. For example, in Market 1.4, there are only positive masses of $(H, \mathcal{H})$, $(H, \hbar)$, $(H, \mathcal{L})$, and (H, ℓ) couples. Under this circumstance, stable marriage payoff differences cannot be uniquely pinned down. $v_{mH} - v_{mL}$ can take a set of values. Since there is no positive mass of $(H, \mathcal{L})$, (H, ℓ), $(L, \mathcal{H})$, and $(L, \hbar)$ couples, inequalities are established from the no blocking pair condition. Scenario 2 for men and scenarios 2 and 3 for women described above might occur.

To derive $v_{mH} - v_{mL}$, we rely on all of these equalities and inequalities. $v_{mH} + v_{w\mathcal{H}} = s_{H\mathcal{H}}$ and $v_{mL} + v_{w\mathcal{H}} \geq s_{L\mathcal{H}}$ imply

$$v_{mH} - v_{mL} \leq (s_{H\mathcal{H}} - v_{w\mathcal{H}}) - (s_{L\mathcal{H}} - v_{w\mathcal{H}}) = s_{H\mathcal{H}} - s_{L\mathcal{H}}.$$

$v_{mH} + v_{w\hbar} \geq s_{H\hbar}$ and $v_{mL} + v_{w\hbar} \geq s_{L\hbar}$ imply

$$v_{mH} - v_{mL} \leq (s_{H\hbar} - v_{w\hbar}) - (s_{L\hbar} - v_{w\hbar}) = s_{H\hbar} - s_{L\hbar}.$$

$v_{mH} + v_{w\mathcal{L}} \geq s_{H\mathcal{L}}$ and $v_{mL} + v_{w\mathcal{L}} \geq s_{L\mathcal{L}}$ imply

$$v_{mH} - v_{mL} \geq (s_{H\mathcal{L}} - v_{w\mathcal{L}}) - (s_{L\mathcal{L}} - v_{w\mathcal{L}}) = s_{H\mathcal{L}} - s_{L\mathcal{L}}.$$

$v_{mH} + v_{w\mathcal{L}} \geq s_{H\mathcal{L}}$ and $v_{mH} + v_{w\ell} = s_{H\ell}$ imply

$$v_{mH} - v_{mL} \geq (s_{H\ell} - v_{w\ell}) - (s_{L\ell} - v_{w\ell}) = s_{H\ell} - s_{L\ell}.$$

Because of income-fitness supermodularity, $s_{H\mathcal{H}} - s_{L\mathcal{H}} \geq s_{H\hbar} - s_{L\hbar}$ and $s_{H\mathcal{L}} - s_{L\mathcal{L}} \geq s_{H\ell} - s_{L\ell}$. Altogether,

$$s_{H\mathcal{L}} - s_{L\mathcal{L}} \leq v_{mH} - v_{mL} \leq s_{H\hbar} - s_{L\hbar}.$$

Therefore,

$$v_{mH} - v_{mL} = (1 - \lambda)(s_{H\hbar} - s_{L\hbar}) + \lambda(s_{H\mathcal{L}} - s_{L\mathcal{L}}),$$

where $\lambda \in [0, 1]$. Three remaining marriage payoff differences of women's follow. Because $v_{mH} + v_{w\mathcal{H}} = s_{H\mathcal{H}}$ and $v_{mL} + v_{w\mathcal{L}} = s_{L\mathcal{L}}$,

$$v_{w\mathcal{H}} - v_{w\mathcal{L}} = s_{H\mathcal{H}} - s_{L\mathcal{L}} - (v_{mH} - v_{mL}).$$

Similarly, because $v_{mH} + v_{w\hbar} = s_{H\hbar}$ and $v_{mL} + v_{w\mathcal{L}} = s_{L\mathcal{L}}$,

$$v_{w\hbar} - v_{w\mathcal{L}} = s_{H\hbar} - s_{L\mathcal{L}} - (v_{mH} - v_{mL}).$$

Finally, since both ℓ and $\mathcal{L}$ women marry L men,

$$v_{w\ell} - v_{w\mathcal{L}} = (s_{L\ell} - v_{mL}) - (s_{L\mathcal{L}} - v_{mL}) = s_{L\ell} - s_{L\mathcal{L}}.$$

Although the four stable marriage payoff differences may not be uniquely determined, equilibrium marriage payoff differences are uniquely determined, which leads to the unique determination of equilibrium investment strategies. In Market 1.5, since $v_{mH} - v_{mL}$ is uniquely determined to be $s_{H\mathcal{L}} - s_{L\mathcal{L}}$, the men's optimal investments are characterized by the cutoff cost $c_m^* = p_m(s_{H\mathcal{L}} - s_{L\mathcal{L}}) + p_m(y_{m\text{H}} - y_{m\text{L}})$. Since $v_{w\mathcal{H}} - v_{w\mathcal{L}}$ is uniquely determined to be $s_{H\mathcal{L}} - s_{L\mathcal{L}}$, the cutoff cost c_{w1}^* is uniquely determined to be $p_w(s_{H\mathcal{H}} - s_{L\mathcal{L}}) + p_w(y_{w\text{H}} - y_{w\text{L}})$. c_{w2}^* is also uniquely pinned down. In the even-numbered Markets where the stable marriage payoff differences are not uniquely determined, investment cost cutoffs can potentially take a range of values. Nonetheless, the equilibrium marriage payoff differences are pinned down uniquely in these cases, because the equilibrium payoff differences will be uniquely determined for the optimal investment strategies: only for a certain unique set of marriage payoff differences, the optimal investment strategies induce marriage-type distributions in the marriage market that features that particular set of marriage payoff differences in the stable outcome.

E. Equilibrium Existence, Uniqueness, and Efficiency

THEOREM 1: *An equilibrium $(\sigma_m^*, \sigma_w^*, G_m^*, G_w^*, G^*, v_m^*, v_w^*)$ always exists. Marriage payoffs are uniquely determined up to a constant and other equilibrium components are uniquely determined.*

PROOF:

Define marriage type distributions as functions of x,

$$\begin{aligned}
&G_{mH}(x) \equiv F_m[p_m(y_{m\text{H}} - y_{m\text{L}}) + p_m(v_{mH} - v_{mL})_x]p_m(2 - p_m),\\
&G_{w\mathcal{H}}(x) \equiv F_w[p_w(y_{w\text{H}} - y_{w\text{L}}) + p_w(v_{w\mathcal{H}} - v_{w\mathcal{L}})_x]p_w +\\
&\quad F_w[p_w(y_{w\text{H}} - y_{w\text{L}}) + rp_w(v_{w\mathcal{H}} - v_{w\mathcal{L}})_x + (1-r)p_w(v_{w\hbar} - v_{w\mathcal{L}})_x\\
&\quad +(1-r)(1-p_w)(v_{w\ell} - v_{w\mathcal{L}})_x](1-p_w)p_w r,\\
&G_{w\mathcal{H}}(x) + G_{w\hbar}(x) \equiv F_w[p_w(y_{w\text{H}} - y_{w\text{L}}) + p_w(v_{w\mathcal{H}} - v_{w\mathcal{L}})_x]p_w +\\
&\quad F_w[p_w(y_{w\text{H}} - y_{w\text{L}}) + rp_w(v_{w\mathcal{H}} - v_{w\mathcal{L}})_x + (1-r)p_w(v_{w\hbar} - v_{w\mathcal{L}})_x\\
&\quad +(1-r)(1-p_w)(v_{w\ell} - v_{w\mathcal{L}})_x](1-p_w)p_w,\\
&G_{w\mathcal{H}}(x) + G_{w\mathcal{L}}(x) \equiv 1 - F_w[p_w(y_{w\text{H}} - y_{w\text{L}}) + rp_w(v_{w\mathcal{H}} - v_{w\mathcal{L}})_x\\
&\quad +(1-r)p_w(v_{w\hbar} - v_{w\mathcal{L}})_x + (1-r)(1-p_w)(v_{w\ell} - v_{w\mathcal{L}})_x](1-p_w)(1-r),\\
&G_{w\mathcal{H}}(x) + G_{w\hbar}(x) + G_{w\mathcal{L}}(x) \equiv 1 - F_w[p_w(y_{w\text{H}} - y_{w\text{L}}) + rp_w(v_{w\mathcal{H}} - v_{w\mathcal{L}})_x\\
&\quad +(1-r)p_w(v_{w\hbar} - v_{w\mathcal{L}})_x + (1-r)(1-p_w)(v_{w\ell} - v_{w\mathcal{L}})_x](1-p_w)^2(1-r).
\end{aligned}$$

The marriage payoff difference functions defined in the text and the marriage type distribution functions defined above vary by x. The directions of changes are laid out in Tables 3 and 4. They will be useful for the proof of equilibrium uniqueness.

TABLE 3—DIRECTIONS OF CHANGES IN MARRIAGE PAYOFF DIFFERENCE AND MARRIAGE TYPE DISTRIBUTION FUNCTIONS AS x INCREASES: $s_{H\hbar} + s_{L\mathcal{L}} > s_{H\mathcal{L}} + s_{L\hbar}$.

	$s_{H\hbar} + s_{L\mathcal{L}} > s_{H\mathcal{L}} + s_{L\hbar}$	
	$x \leq x_{1.5}$	$x \geq x_{1.5}$
$(v_{mH} - v_{mL})_x$	↓	↓
$(v_{w\mathcal{H}} - v_{w\mathcal{L}})_x$	↑	-
$(v_{w\hbar} - v_{w\mathcal{L}})_x$	↑	-
$(v_{w\ell} - v_{w\mathcal{L}})_x$	-	↓
$G_{mH}(x)$	↓	↓
$G_{w\mathcal{H}}(x)$	↑	↓
$G_{w\mathcal{H}}(x) + G_{w\hbar}(x)$	↑	↓
$(G_{w\mathcal{H}} + G_{w\hbar} + G_{w\mathcal{L}})(x)$	↓	↑

First, suppose $s_{H\hbar} + s_{L\mathcal{L}} > s_{H\mathcal{L}} + s_{L\hbar}$. Define

$$\phi_1(x) \equiv \begin{cases} 0, & G_{mH}(x) < G_{w\mathcal{H}}(x) \\ [0, x_{1.3}], & G_{mH}(x) = G_{w\mathcal{H}}(x) \\ x_{1.3}, & G_{w\mathcal{H}}(x) < G_{mH}(x) < G_{w\mathcal{H}}(x) + G_{w\hbar}(x) \\ [x_{1.3}, x_{1.5}], & G_{mH}(x) = G_{w\mathcal{H}}(x) + G_{w\hbar}(x) \\ x_{1.5}, & (G_{w\mathcal{H}} + G_{w\hbar})(x) < G_{mH}(x) < (G_{w\mathcal{H}} + G_{w\hbar} + G_{w\mathcal{L}})(x) \\ [x_{1.5}, 1], & G_{mH}(x) = G_{w\mathcal{H}}(x) + G_{w\hbar}(x) + G_{w\mathcal{L}}(x) \\ 1, & G_{mH}(x) > G_{w\mathcal{H}}(x) + G_{w\hbar}(x) + G_{w\mathcal{L}}(x) \end{cases}$$

where indices $x_{1.3}$ and $x_{1.5}$ represent the stable outcome in Markets 1.3 and 1.5. ϕ_1 is upper-hemicontinuous and $\phi_1(x)$ is convex for all $x \in [0,1]$. Therefore, ϕ_1 has a fixed point, and an equilibrium exists. Equilibrium uniqueness is shown as follows.

Case 1a. $G_{mH}(x_{1.5}) < (G_{w\mathcal{H}} + G_{w\hbar})(x_{1.5}) < (G_{w\mathcal{H}} + G_{w\hbar} + G_{w\mathcal{L}})(x_{1.5})$**.**
$\phi_1(x_{1.5}) \subseteq [0, x_{1.3}]$. For any $x < x_{1.5}$, $G_{mH}(x)$ decreases, and $(G_{w\mathcal{H}} + G_{w\hbar})(x)$ and $G_{w\mathcal{H}}(x)$ increase, so $\phi_1(x)$ decreases, and there is a unique fixed point $x^* \leq x_{1.5}$. For any $x > x_{1.5}$, $G_{mH}(x) < G_{mH}(x_{1.5}) < (G_{w\mathcal{H}} + G_{w\hbar} + G_{w\ell})(x_{1.5}) < (G_{w\mathcal{H}} + G_{w\hbar} + G_{w\ell})(x)$, so $x' \in \phi_1(x)$ implies $x \leq x_{1.5}$. Hence there is no fixed point $x^* > x_{1.5}$.

Case 1b. $G_{w\mathcal{H}} + G_{w\hbar})(x_{1.5}) \leq G_{mH}(x_{1.5}) \leq (G_{w\mathcal{H}} + G_{w\hbar} + G_{w\mathcal{L}})(x_{1.5})$**.**
$x^* = x_{1.5}$ is a fixed point. For any $x < x_{1.5}$, $G_{mH}(x) \geq G_{mH}(x_{1.5}) \geq (G_{w\mathcal{H}} + G_{w\hbar})(x_{1.5}) \geq (G_{w\mathcal{H}} + G_{w\hbar})(x)$,so $x' \in \phi_1(x)$ implies $x' \geq x_{1.5}$. For any $x > x_{1.5}$, $G_{mH}(x) \leq G_{mH}(x_{1.5}) \leq (G_{w\mathcal{H}} + G_{w\hbar} + G_{w\mathcal{L}}(x_{1.5}) \leq (G_{w\mathcal{H}} + G_{w\hbar} + G_{w\mathcal{L}}(x)$, so $x' \in \phi_1(x)$ implies $x' \leq x_{1.5}$.

Case 1c. $G_{mH}(x_{1.5}) > (G_{w\mathcal{H}} + G_{w\hbar} + G_{w\mathcal{L}})(x_{1.5}) > (G_{w\mathcal{H}} + G_{w\hbar})(x_{1.5})$.
$\phi_1(x_{1.5}) = 1$. For any $x \leq x_{1.5}$, $G_{mH}(x) \geq G_{mH}(x_{1.5}) > (G_{w\mathcal{H}} + G_{w\hbar})(x_{1.5}) \geq (G_{w\mathcal{H}} + G_{w\hbar})(x)$, so $\phi_1(x) = 1$. For any $x > x_{1.5}$, $G_{mH}(x)$ strictly decreases and $(G_{mH} + G_{w\hbar} + G_{w\mathcal{L}})(x)$ strictly increases, so $\phi_1(x)$ weakly decreases and there's a unique fixed point $x^* \geq x_{1.5}$.

TABLE 4—DIRECTIONS OF CHANGES IN MARRIAGE PAYOFF DIFFERENCE AND MARRIAGE TYPE DISTRIBUTION FUNCTIONS AS x INCREASES: $s_{H\hbar} + s_{L\mathcal{L}} < s_{H\mathcal{L}} + s_{L\hbar}$.

	$s_{H\hbar} + s_{L\mathcal{L}} < s_{H\mathcal{L}} + s_{L\hbar}$		
	$x \leq x_{2.3}$	$x_{2.3} \leq x \leq x_{2.5}$	$x \geq x_{2.5}$
$(v_{mH} - v_{mL})_x$	↓	↓	↓
$(v_{w\mathcal{H}} - v_{w\mathcal{L}})_x$	↑	-	-
$(v_{w\hbar} - v_{w\mathcal{L}})_x$	-	↓	-
$(v_{w\ell} - v_{w\mathcal{L}})_x$	-	↓	↓
$G_{mH}(x)$	↓	↓	↓
$G_{w\mathcal{H}}(x)$	↑	↓	↓
$G_{w\mathcal{H}}(x) + G_{w\mathcal{L}}(x)$	↓	↑	↑
$(G_{w\mathcal{H}} + G_{w\hbar} + G_{w\mathcal{L}})(x)$	↓	↑	↑

Analogously, suppose $s_{H\hbar} + s_{L\mathcal{L}} \leq s_{H\mathcal{L}} + s_{L\hbar}$. Define

$$\phi_2(x) \equiv \begin{cases} 0, & G_{mH}(x) < G_{w\mathcal{H}}(x) \\ [0, x_{2.3}], & G_{mH}(x) = G_{w\mathcal{H}}(x) \\ x_{2.3}, & G_{w\mathcal{H}}(x) < G_{mH}(x) < G_{w\mathcal{H}}(x) + G_{w\mathcal{L}}(x) \\ [x_{2.3}, x_{2.5}], & G_{mH}(x) = G_{w\mathcal{H}}(x) + G_{w\mathcal{L}}(x) \\ x_{2.5}, & (G_{w\mathcal{H}} + G_{w\mathcal{L}})(x) < G_{mH}(x) < (G_{w\mathcal{H}} + G_{w\mathcal{L}} + G_{w\hbar})(x) \\ [x_{2.5}, 1], & G_{mH}(x) = G_{w\mathcal{H}}(x) + G_{w\mathcal{L}}(x) + G_{w\hbar}(x) \\ 1, & G_{mH}(x) > G_{w\mathcal{H}}(x) + G_{w\mathcal{L}}(x) + G_{w\hbar}(x) \end{cases}$$

where indices $x_{2.3}$ and $x_{2.5}$ represent the stable outcome in Markets 2.3 and 2.5. ϕ_2 is upper-hemicontinuous and $\phi_2(x)$ is convex for all x. Therefore, ϕ_2 has a fixed point, and an equilibrium exists. Equilibrium uniqueness is shown as follows.
Case 2a. $G_{mH}(x_{2.3}) < G_{w\mathcal{H}}(x_{2.3}) < (G_{w\mathcal{H}} + G_{w\mathcal{L}})(x_{2.3})$.
$\phi_2(x_{2.3}) = 0$. For any $x \leq x_{2.3}$, $G_{mH}(x)$ decreases and $G_{w\mathcal{H}}(x)$ increases, so $\phi_2(x)$ decreases and there is a unique fixed point $x^* \leq x_{2.3}$. For any $x > x_{2.3}$, $G_{mH}(x) \leq G_{mH}(x_{2.3}) < (G_{w\mathcal{H}} + G_{w\mathcal{L}})(x_{2.3}) \leq (G_{w\mathcal{H}} + G_{w\mathcal{L}})(x)$, so $x' \in \phi_2(x)$ implies $x' \leq x_{2.3}$.
Case 2b. $G_{w\mathcal{H}}(x_{2.3}) \leq G_{mH}(x_{2.3}) \leq (G_{w\mathcal{H}} + G_{w\mathcal{L}})(x_{2.3})$.
$x^* = x_{2.3}$ is a fixed point. For any $x < x_{2.3}$, $G_{w\mathcal{H}}(x) \leq G_{w\mathcal{H}}(x_{2.3}) \leq G_{mH}(x_{2.3}) \leq G_{mH}(x)$, so $x' \in \phi_2(x)$ implies $x' \geq x_{2.3}$. For any $x > x_{2.3}$, $G_{mH}(x) \leq G_{mH}(x_{2.3}) \leq (G_{w\mathcal{H}} + G_{w\mathcal{L}})(x_{2.3}) \leq (G_{w\mathcal{H}} + G_{w\mathcal{L}})(x)$, so $x' \in \phi_2(x)$ implies $x' \leq x_{2.3}$. Hence,

$x^* = x_{2.3}$ is the unique fixed point.

Case 2c. $G_{w\mathcal{H}}(x_{2.3}) < (G_{w\mathcal{H}} + G_{w\mathcal{L}})(x_{2.3}) < G_{mH}(x_{2.3})$**.**
$\phi_2(x_{2.3}) \subseteq [x_{2.5}, 1]$. For any $x < x_{2.3}$, $G_{w\mathcal{H}}(x) \leq G_{w\mathcal{H}}(x_{2.3}) \leq G_{mH}(x_{2.3}) \leq G_{mH}(x)$, $x' \in \phi_2(x)$ implies $x' \geq x_{2.3}$. For any $x \geq x_{2.3}$, $(G_{w\mathcal{H}} + G_{w\mathcal{L}})(x)$ and $(G_{w\mathcal{H}} + G_{w\mathcal{L}} + G_{w\hbar})(x)$ increase and $G_{mH}(x)$ decreases in x, so $\phi_2(x)$ decreases, and there's a unique fixed point $x^* \geq x_{2.3}$.

Namely, the equilibrium can be characterized by a single x^*. If $s_{H\hbar} + s_{L\mathcal{L}} > s_{H\mathcal{L}} + s_{L\hbar}$, equilibrium x^* is characterized as follows.

1.1 $G_{mH}(0) < G_{w\mathcal{H}}(0)$: $x^* = 0$.

1.2 $G_{mH}(0) \geq G_{w\mathcal{H}}(0)$ and $G_{mH}(x_{1.3}) < G_{w\mathcal{H}}(x_{1.3})$: x^* is determined by $G_{mH}(x^*) = G_{w\mathcal{H}}(x^*)$.

1.3 $G_{w\mathcal{H}}(x_{1.3}) \leq G_{mH}(x_{1.3}) < G_{w\mathcal{H}}(x_{1.3}) + G_{w\hbar}(x_{1.3})$: $x^* = x_{1.3}$.

1.4 $G_{mH}(x_{1.3}) \geq (G_{w\mathcal{H}} + G_{w\hbar})(x_{1.3})$ and $G_{mH}(x_{1.5}) < (G_{w\mathcal{H}} + G_{w\hbar})(x_{1.5})$: x^* is determined by $G_{mH}(x^*) = G_{w\mathcal{H}}(x^*) + G_{w\hbar}(x^*)$.

1.5 $G_{w\mathcal{H}}(x_{1.5}) + G_{w\hbar}(x_{1.5}) \leq G_{mH}(x_{1.5}) < G_{w\mathcal{H}}(x_{1.5}) + G_{w\hbar}(x_{1.5}) + G_{w\mathcal{L}}(x_{1.5})$: $x^* = x_{1.5}$.

1.6 $G_{mH}(x_{1.5}) \geq G_{w\mathcal{H}}(x_{1.5}) + G_{w\hbar}(x_{1.5}) + G_{w\mathcal{L}}(x_{1.5})$ and $G_{mH}(1) < G_{w\mathcal{H}}(1) + G_{w\hbar}(1) + G_{w\mathcal{L}}(1)$: x^* is determined by $G_{mH}(x^*) = G_{w\mathcal{H}}(x^*) + G_{w\hbar}(x^*) + G_{w\mathcal{L}}(x^*)$.

1.7 $G_{mH}(1) \geq G_{w\mathcal{H}}(1) + G_{w\hbar}(1) + G_{w\mathcal{L}}(1)$: $x^* = 1$.

If $s_{H\hbar} + s_{L\mathcal{L}} \leq s_{H\mathcal{L}} + s_{L\hbar}$, equilibrium x^* is characterized as follows.

2.1 $G_{mH}(0) < G_{w\mathcal{H}}(0)$: $x^* = 0$.

2.2 $G_{mH}(0) \geq G_{w\mathcal{H}}(0)$ and $G_{mH}(x_{2.3}) < G_{w\mathcal{H}}(x_{2.3})$: x^* is determined by $G_{mH}(x^*) = G_{w\mathcal{H}}(x^*)$.

2.3 $G_{w\mathcal{H}}(x_{2.3}) \leq G_{mH}(x_{2.3}) < G_{w\mathcal{H}}(x_{2.3}) + G_{w\mathcal{L}}(x_{2.3})$: $x^* = x_{2.3}$.

2.4 $G_{mH}(x_{2.3}) \geq (G_{w\mathcal{H}} + G_{w\mathcal{L}})(x_{2.3})$ and $G_{mH}(x_{2.5}) < (G_{w\mathcal{H}} + G_{w\mathcal{L}})(x_{2.5})$: x^* is determined by $G_{mH}(x^*) = G_{w\mathcal{H}}(x^*) + G_{w\mathcal{L}}(x^*)$.

2.5 $G_{w\mathcal{H}}(x_{2.5}) + G_{w\mathcal{L}}(x_{2.5}) \leq G_{mH}(x_{2.5}) < G_{w\mathcal{H}}(x_{2.5}) + G_{w\mathcal{L}}(x_{2.5}) + G_{w\hbar}(x_{2.5})$: $x^* = x_{2.5}$.

2.6 $G_{mH}(x_{2.5}) \geq G_{w\mathcal{H}}(x_{2.5}) + G_{w\mathcal{L}}(x_{2.5}) + G_{w\hbar}(x_{2.5})$ and $G_{mH}(1) < G_{w\mathcal{H}}(1) + G_{w\mathcal{L}}(1) + G_{w\hbar}(1)$: x^* is determined by $G_{mH}(x^*) = G_{w\mathcal{H}}(x^*) + G_{w\mathcal{L}}(x^*) + G_{w\hbar}(x^*)$.

2.7 $G_{mH}(1) \geq G_{w\mathcal{H}}(1) + G_{w\mathcal{L}}(1) + G_{w\hbar}(1)$: $x^* = 1$.

QED

Furthermore, equilibrium investments are socially efficient. The result has been observed in many investments-and-matching models (Cole, Mailath and Postlewaite, 2001; Peters and Siow, 2002; Iyigun and Walsh, 2007; Chiappori, Iyigun and Weiss, 2009; Dizdar, 2013; Nöldeke and Samuelson, 2015). The key is that the private marginal marriage payoff gain from investing is also the social gain from investing, which is a feature of the competitive matching model (Gretsky, Ostroy and Zame, 1999). However, the result is not a corollary of previous results because there is uncertainty with the investments' outcome. In previous papers, private deterministic investments can be thought to be contracted before they are made, so socially efficient contracts on investments can always be signed and reinforced between pairs who will eventually match (Cole, Mailath and Postlewaite, 2001; Dizdar, 2013), but in this model, because of uncertainty, no such pairwise contract is available. Nonetheless, uniqueness of the equilibrium and agents' rational expectations of their private payoffs guarantee that there is no coordination failure in their investment choices.

THEOREM 2: *Equilibrium investments are socially efficient.*

PROOF:

It is socially more efficient for a lower cost person to invest. Therefore, the socially efficient investments can be characterized by cutoffs (c_{m1}, c_{m2}, c_{w1}, c_{w2}), where men below cost c_{m1} and women below cost c_{w2} invest in college and men below cost c_{m2} and women below cost c_{w2} invest in career. The resulting stationary marriage-type and income distributions are

$$G_{mH}(c_{m1}, c_{m2}) = F_m(c_{m1})p_m + F_m(c_{m2})(1 - p_m)p_m$$

and $G_{w\mathcal{H}}(c_{w1}, c_{w2})$, $G_{w\hbar}(c_{w1}, c_{w2})$, $G_{w\mathcal{L}}(c_{w1}, c_{w2})$ as described in equations (6), (7), and (8). The total net surplus $V(c_{m1}, c_{m2}, c_{w1}, c_{w2})$ is

$$\begin{aligned}
&y_{m\mathrm{L}} + G_{mH}(c_{m1}, c_{m2})(y_{m\mathrm{H}} - y_{m\mathrm{L}}) \\
&+y_{w\mathrm{L}} + (G_{w\mathcal{H}}(c_{w1}, c_{w2}) + G_{w\hbar}(c_{w1}, c_{w2}))(y_{w\mathrm{H}} - y_{w\mathrm{L}}) \\
&+s(G_{mH}(c_{m1}, c_{m2}), G_{w\mathcal{H}}(c_{w1}, c_{w2}), G_{w\hbar}(c_{w1}, c_{w2}), G_{w\mathcal{L}}(c_{w1}, c_{w2})) \\
&-F_m(c_{m1})c_m - F_m(c_{m2})(1 - p_m)c_m - F_w(c_{w1})c_w - F_w(c_{w2})(1 - p_w)c_w
\end{aligned}$$

where $s(G_{mH}, G_{w\mathcal{H}}, G_{w\hbar}, G_{w\mathcal{L}})$ is the maximal marriage surplus when the marriage market has marriage-type distributions $G_m = (G_{mH}, 1-G_{mH})$ and $G_w = (G_{w\mathcal{H}}, G_{w\hbar}, G_{w\mathcal{L}}, 1-G_{w\mathcal{H}} - G_{w\hbar} - G_{w\mathcal{L}})$. In Gretsky, Ostroy and Zame (1999), $s(\cdot)$ is called the social gains function. It is continuous but not necessarily differentiable. However, the sub-differential ∂s corresponds to the set of stable marriage payoffs. Therefore, the sub-differential of $s(\cdot)$ with respect to G_{mH} is $\partial s/\partial G_{mH} - \partial s/\partial G_{mL} = v_{mH} - v_{mL}$, the set of stable marriage payoff differences. The sub-differential of V with respect to c_{m1} is

$$\frac{\partial V}{\partial c_{m1}} = \frac{\partial G_{mH}(c_{m1}, c_{m2})}{\partial c_{m1}}(y_{m\mathrm{H}} - y_{m\mathrm{L}}) + \frac{\partial s}{\partial G_{mH}} \frac{\partial G_{mH}(c_{m1}, c_{m2})}{\partial c_{m1}} - f_m(c_{m1})c_m,$$

which simplifies to

$$\frac{\partial V}{\partial c_{m1}} = f_m(c_{m1})[p_m(y_{m\mathrm{H}} - y_{m\mathrm{L}}) + p_m(v_{mH} - v_{mL}) - c_m].$$

Similarly,

$$\begin{aligned}\frac{\partial V}{\partial c_{m2}} &= f_m(c_{m2})[p_m(y_{m\mathrm{H}} - y_{m\mathrm{L}}) + p_m(v_{mH} - v_{mL}) - c_m].\\ \frac{\partial V}{\partial c_{w1}} &= f_m(c_{w1})[p_w(y_{w\mathrm{H}} - y_{w\mathrm{L}}) + p_m(v_{w\mathcal{H}} - v_{w\mathcal{L}}) - c_w].\\ \frac{\partial V}{\partial c_{w2}} &= f_m(c_{w2})[p_w(y_{w\mathrm{H}} - y_{w\mathrm{L}}) + p_w(v_{w\mathcal{H}} - v_{w\mathcal{L}})\\ &\quad +(1-r)p_w(v_{w\hbar} - v_{w\mathcal{L}}) + (1-r)(1-p_w)(v_{w\ell} - v_{w\mathcal{L}}) - c_w].\end{aligned}$$

$0 \in \frac{\partial V}{\partial c_{m1}}, \frac{\partial V}{\partial c_{m2}}, \frac{\partial V}{\partial c_{w1}}, \frac{\partial V}{\partial c_{w2}}$ if $c_{m1} = c_{m2} = c_m^*, c_{w1} = c_{w1}^*, c_{w2} = c_{w2}^*$, by the equilibrium conditions. And $0 \in \frac{\partial V}{\partial c_{m1}}, \frac{\partial V}{\partial c_{m2}}, \frac{\partial V}{\partial c_{w1}}, \frac{\partial V}{\partial c_{w2}}$ only if $c_{m1} = c_{m2} = c_m^*, c_{w1} = c_{w1}^*, c_{w2} = c_{w2}^*$ by the uniqueness of the equilibrium. Therefore, the equilibrium investments are socially efficient.
QED

III. Shrinking College Gender Gap

PROPOSITION 1: *Suppose there are strictly fewer high-income women than high-income men before and after the changes.*[3] *Women's college enrollment $F_w(c_{w1}^*)$ increases and the college gender gap $F_m(c_m^*) - F_w(c_{w1}^*)$ shrinks when*

- *(a) (societal changes) women's investment cost distribution F_w decreases to a first-order stochastically dominated distribution F_w',*
- *(b) (labor-market changes) women's probability of success p_w and women's labor market gain $(y_{w\mathrm{H}} - y_{w\mathrm{L}})$ increase, and*
- *(c) (marriage-market changes) a high-income man's marriage surplus gain from having a high-income wife relative to a low-income wife, $s_{H\mathcal{H}} - s_{H\mathcal{L}}$, the degree of positive-assortative matching in incomes, increases.*

PROOF:
The fact that there are strictly fewer high-income women than high-income men in equilibrium means $G_{w\mathcal{H}}^* + G_{w\hbar}^* < G_{mH}^*$. The equilibrium marriage market must be as one of the following (illustrated in Figure 2): Markets 1.5-1.7 and 2.3-2.7. In these Markets, women's equilibrium marriage-payoff difference is

$$v_{w\mathcal{H}}^* - v_{w\mathcal{L}}^* = s_{H\mathcal{H}} - s_{H\mathcal{L}}.$$

[3] The remarks below discuss the necessity of this condition.

This result can be seen from stable matching and stability conditions (or directly from Table 2). Since there are strictly fewer high-income women than high-income men in the marriage market and there is positive-assortative matching by incomes and by income and reproductive fitness (by Lemmas 3a and 3b), there are always positive masses of $(H, \mathcal{H})$ and $(H, \mathcal{L})$ couples. Thus two pairwise efficiency conditions are satisfied:

$$\begin{aligned} v^*_{mH} + v^*_{w\mathcal{H}} &= s_{H\mathcal{H}}, \\ v^*_{mH} + v^*_{w\mathcal{L}} &= s_{H\mathcal{L}}. \end{aligned}$$

Subtracting the two stability conditions from each other yields the desired result $v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}} = s_{H\mathcal{H}} - s_{H\mathcal{L}}$. Women's college enrollment rate is $F_w(c^*_{w1})$, which can be written in the primitives of the model as,

$$F_w(p_w(y_{w\mathrm{H}} - y_{w\mathrm{L}} + s_{H\mathcal{H}} - s_{H\mathcal{L}})).$$

Clearly, the expression increases when (a) F_w shifts to first-order stochastically dominated F'_w, (b) p_w increases and $y_{w\mathrm{H}} - y_{w\mathrm{L}}$ increases, and/or (d) $s_{H\mathcal{H}} - s_{H\mathcal{L}}$ increases.

On the other hand, men's marriage-market payoff difference is $s_{H\tau_w} - s_{H\tau_w}$ for some $\tau_w \in \{\mathcal{H}, \mathcal{L}, \hbar, \ell\}$, and the mass of college men is $F_m(p_m(y_{m\mathrm{H}} - y_{m\mathrm{L}} + s_{H\tau_w} - s_{H\tau_w}))$. Men's college enrollment is not affected by the changes in the aforementioned factors, when there are strictly fewer high-income women than high-income men (the result may not hold when there is an equal number of high-income men and high-income women). Consequently, the college gender gap $F_m(c_m)^* - F_w(c^*_{w1})$ shrinks.
QED.

REMARKS ABOUT PROPOSITION 1

$G^*_{w\mathcal{H}} + G^*_{w\hbar} < G^*_{mH}$ is necessary for Proposition 1b. In Models 1.2, 1.4, and 2.2, the change in women's college enrollment in response to increase in p_w is indeterminate. Due to the parsimonious specification of the model, p_w denotes not only the success probability of a college investor but also the success probability of a career investor. As a result, when p_w increases, three effects arise (even more complicated than the two effects highlighted when $y_{w\mathrm{H}} - y_{w\mathrm{L}}$ changes): (1) marginal college decisions makers (those with costs right below c^*_{w1}) are incentivized to invest more because of higher expected gain in the labor market and the marriage market, (2) marginal career investors (those with costs right below c^*_{w2}), and (3) more infra-marginal (college and career) investors receive a high-income offer. Effect (1) results in more college investors in equilibrium, but effects (2) and (3) result in fewer college investors in equilibrium as the increase in high-income types in the marriage market drives down the marriage-market returns to investment.

$G^*_{mH} > G^*_{w\mathcal{H}}$ but not $G^*_{mH} > G^*_{w\mathcal{H}} + G^*_{w\hbar}$ is necessary for Proposition 1b. When $G^*_{mH} = G^*_{w\mathcal{H}}$, it is possible that an increase in $y_{w\mathrm{H}} - y_{w\mathrm{L}}$ results in a strict decrease in the mass of women going to college, because of the following argument. In Models 1.2 and 2.2, $G^*_{mH} = G^*_{w\mathcal{H}}$ in equilibrium, and when $y_{w\mathrm{H}} - y_{w\mathrm{L}}$ increases, the total mass of women achieving high-income increases in equilibrium as a result of the increase in

$y_{w\mathrm{H}} - y_{w\mathrm{L}}$, which brings down equilibrium $v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$. There are two effects on women's investments that contribute to the eventual increase in the mass of type H women: a change in college enrollment and a change in career investment. It is thus possible that the increase in career investment is so large that it crowds out and dis-incentivizes the college investors. In equilibrium, the masses of male college investors and female career investors increase but the mass of female college investors decreases. The counterintuitive result partly arises from the overly parsimonious set up of the model. In the model, $y_{w\mathrm{H}} - y_{w\mathrm{L}}$ is not only the income gain when a woman makes a college investment but also the increase in income when a woman makes a career investment. If $y_{w\mathrm{H}} - y_{w\mathrm{L}}$ only denotes the income gain for the college investment and not for the career investment (for example, specify that the income gain after a college investment to be $y_{wH1} - y_{wL1}$ and the income gain after a career investment to be $y_{wH2} - y_{wL2}$), then the possibility that an increase in income gain lowers the investment incentive as in Models 1.2 and 2.2 will not be present.

IV. Main Result: Reversed College Gender Gap and Persistent Gender Pay Gap

A. *Reversed College Gender Gap*

PROPOSITION 2: *Suppose that the investment cost distributions, investment success probabilities, labor market income gain, and the gender roles in the surplus are symmetric ($F_m = F_w$, $p_m = p_w$, $y_{m\mathrm{H}} - y_{m\mathrm{L}} = y_{w\mathrm{H}} - y_{w\mathrm{L}}$, $s_{H\mathcal{L}} = s_{L\mathcal{H}}$) but women continue to have shorter expected reproductive length ($r < 1$). Strictly more women than men go to college in equilibrium.*

PROOF:

Suppose that the setting is gender-symmetric except for reproductive fitness: $F_m = F_w \equiv F$, $p_m = p_w \equiv p$, $y_{m\mathrm{H}} - y_{m\mathrm{L}} = y_{w\mathrm{H}} - y_{w\mathrm{L}} \equiv y_\mathrm{H} - y_\mathrm{L}$, $s_{H\mathcal{L}} = s_{L\mathcal{H}}$, and $r < 1$. The three equilibrium investment cutoffs become $c^*_m = p(y_\mathrm{H} - y_\mathrm{L}) + p(v^*_{mH} - v^*_{mL})$, $c^*_{w1} = p(y_\mathrm{H} - y_\mathrm{L}) + p(v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}})$, and $c^*_{w2} = p(y_\mathrm{L} - y_\mathrm{L}) + p(v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}) - (1 - r)[p(v^*_{w\mathcal{H}} - v^*_{w\hbar}) + (1 - p)(v^*_{w\mathcal{L}} - v^*_{w\ell})]$. The proof is by contradiction. Suppose that weakly fewer women than men go to college in equilibrium, $F(c^*_m) \geq F(c^*_{w1})$. *Claim 1* and *Claim 2* below contradict each other.

Claim 1. $v^*_{mH} - v^*_{mL} \geq v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$. *Proof.* It directly follows from $c^*_m \geq c^*_{w1}$.

Claim 2. $v^*_{mH} - v^*_{mL} < v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$. *Proof.* Since $r < 1$, $c^*_{w1} > c^*_{w2}$. The mass of high-income men, $G^*_{mH} = F(c^*_m)p(2-p)$, is strictly more than the mass of high-income women, $G^*_{w\mathcal{H}} + G^*_{w\hbar} < F(c^*_{w1})p(2-p)$. As a result, there is always a positive mass of (H, L) couples in equilibrium matching. Stability condition 1: $v^*_{mH} + v^*_{w\mathcal{L}} = s_{H\mathcal{L}}$. Two other stability conditions always hold. Stability condition 2: $v^*_{mH} + v^*_{w\mathcal{H}} = s_{H\mathcal{H}}$, because there is always a positive mass of (H, H) couples. Stability condition 3: $v^*_{mL} + v^*_{w\mathcal{L}} \geq s_{L\mathcal{L}}$. From stability conditions 1 and 2: $v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}} = s_{H\mathcal{H}} - s_{H\mathcal{L}}$. From stability conditions 1 and 3: $v^*_{mH} - v^*_{mL} \leq s_{H\mathcal{L}} - s_{L\mathcal{L}} = s_{L\mathcal{H}} - s_{L\mathcal{L}}$, where the equality follows from $s_{H\mathcal{L}} = s_{L\mathcal{H}}$. Due to Assumption 1 strict income-income supermodularity, $v^*_{mH} - v^*_{mL} \leq s_{L\mathcal{H}} - s_{L\mathcal{L}} < s_{H\mathcal{H}} - s_{H\mathcal{L}} = v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$.

QED

B. *Persistent Gender Pay Gap*

We have a bit less general Proposition 3 in this enriched model than in the main text. There are possibly more high-income women than high-income men, but there cannot be more fit high-income women than fit high-income men.

PROPOSITION 3: *Suppose that the setting is the same as in Proposition 2: the investment cost distributions, investment success probabilities, labor market income gain, and the gender roles in the surplus are symmetric ($F_m = F_w$, $p_m = p_w$, $y_{m\mathrm{H}} - y_{m\mathrm{L}} = y_{w\mathrm{H}} - y_{w\mathrm{L}}$, $s_{H\mathcal{L}} = s_{L\mathcal{H}}$) but women have a reproductive cost ($r < 1$). Fewer fit women than men earn a high income in equilibrium, so fit women's average income is lower than men's.*

The proof mimics the proof of Proposition 3 in the main text.
PROOF:

Suppose by contradiction that strictly more fit high-income women than (fit) high-income men earn a high income: $G_{mH} < G_{w\mathcal{H}}$. Therefore, the marriage-type distributions are either of Case 1.1 or Case 2.1. In those cases, the stable marriage payoff differences are $v^*_{mH} - v^*_{mL} = s_{H\mathcal{H}} - s_{L\mathcal{H}}$ and $v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}} = s_{L\mathcal{H}} - s_{L\mathcal{L}}$. Since $s_{H\mathcal{L}} = s_{L\mathcal{H}}$, $v^*_{mH} - v^*_{mL} = s_{H\mathcal{H}} - s_{L\mathcal{H}} = s_{H\mathcal{H}} - s_{H\mathcal{L}} > s_{L\mathcal{H}} - s_{L\mathcal{L}} = v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$. However, if $v^*_{mH} - v^*_{mL} > v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$, then there should be strictly more high-income men than high-income women, as $G_{mH} = F(p_m\Delta y + p\Delta v^*_m)p(2-p) > F(p_m\Delta y + p\Delta v^*_w)[p+p(1-p)] > F(p_m\Delta y + p\Delta v^*_w)p + F(p_m\Delta y + p\Delta v^*_w - k^*)p(1-p) = G_{w\mathcal{H}}$, where k^* is the endogenous fertility cost. The conclusion that $G_{mH} > G_{w\mathcal{H}}$ contradicts with the premise that $G_{mH} < G_{w\mathcal{H}}$, so we must have $G_{mH} \geq G_{w\mathcal{H}}$.

C. *Marginal versus Average Returns to College*

The implications about marginal and average returns on college in the enriched model are the same as in the basic model. The expressions of the return on college are the same for all agents except for college-and-career-investing women. A college-and-career investing woman's equilibrium return on college is

$$p_w\Delta y_w + p_w\Delta^*_w + (1-p_w)[p_w(\Delta y_w + \Delta^*_w) - c_w] \\ - (1-p_w)(1-r)[p_w\Delta^*_w + (1-p_w)(v^*_{w\mathcal{L}} - v^*_{w\ell})].$$

Proposition 4 about the implications of net returns on college is the same as in the main text. Figures 6 and 7 are the same as in the main text.

PROPOSITION 4: *The following statements regarding the equilibrium net returns on college hold.*

(a) The equilibrium net returns on college differ by gender and by ability within a gender.

(b) The equilibrium net returns on college of marginal *college-investing women are* always *higher than those of the men of the same abilities.*

(c) *The* average *equilibrium net returns on college of women are could be higher or lower than those of men, even though more women than men go to college in equilibrium.*

D. *Key Assumptions Underlying the Main Result*

Subsequently I derive whether more men or women go to college under the three alternative surplus modularity assumptions: (1) income-income submodularity and income-fitness supermodularity, (2) income-income supermodularity and income-fitness submodularity, and (3) income-income submodularity and income-fitness submodularity. In these three alternative configurations, more women than men may go to college when income-fitness supermodularity holds even when income-income supermodularity fails. Hence, the robustness checks show that income-income supermodularity can be relaxed in more general settings than the setting in the basic model in the main text, but some supermodularity is still needed to generate the reversed college gender gap result.

INCOME-INCOME SUBMODULARITY AND INCOME-FITNESS SUPERMODULARITY

Figure 3 shows the 14 possible matchings under income-income submodularity and income-fitness supermodularity and Table 5 shows the stable payoff differences $v_{mH} - v_{mL}$ and $v_{w\mathcal{H}} - v_{w\mathcal{L}}$ under the 14 different markets. In all the markets, $v^*_{mH} - v_{mL} = v^*_{w\mathcal{H}} - v_{w\mathcal{L}}$ or $v^*_{mH} - v_{mL} < v^*_{w\mathcal{H}} - v_{w\mathcal{L}}$, so regardless of the equilibrium, weakly more women than men go to college.

INCOME-INCOME SUPERMODULARITY AND INCOME-FITNESS SUBMODULARITY

Figure 4 shows the 14 possible stable matchings under income-income supermodularity and income-fitness submodularity, and Table 6 shows the stable payoff differences in these 14 markets. When the setting is gender-symmetric except for fitness, Markets 1.1-1.4 and 2.1-2.2 will not appear in equilibrium, because $v^*_{mH} - v^*_{mL} > v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$ implies $c^*_m > c^*_{w1} > c^*_{w2}$ and $G^*_{mH} > G^*_{w\mathcal{H}} + G^*_{w\hbar}$. In the rest of the possible markets, $v^*_{mH} - v^*_{mL} > v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$, $v^*_{mH} - v^*_{mL} = v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$, or $v^*_{mH} - v^*_{mL} < v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$. Therefore, the sign of the college gender gap is ambiguous.

INCOME-INCOME SUBMODULARITY AND INCOME-FITNESS SUBMODULARITY

Figure 5 shows the 14 possible stable matchings under income-income submodularity and income-fitness submodularity and Table shows the stable marriage payoff differences in the 14 markets. When $s_{H\mathcal{L}} = s_{L\mathcal{H}}$, in all markets, men's payoff difference is weakly greater than women's. Therefore, in equilibrium, weakly more men than women go to college.

V. Widening College Gender Gap?

PROPOSITION 5: *Women's college enrollment rate weakly decreases as the fitness probability r increases. When the setting is completely gender-symmetric ($F_m = F_w$,*

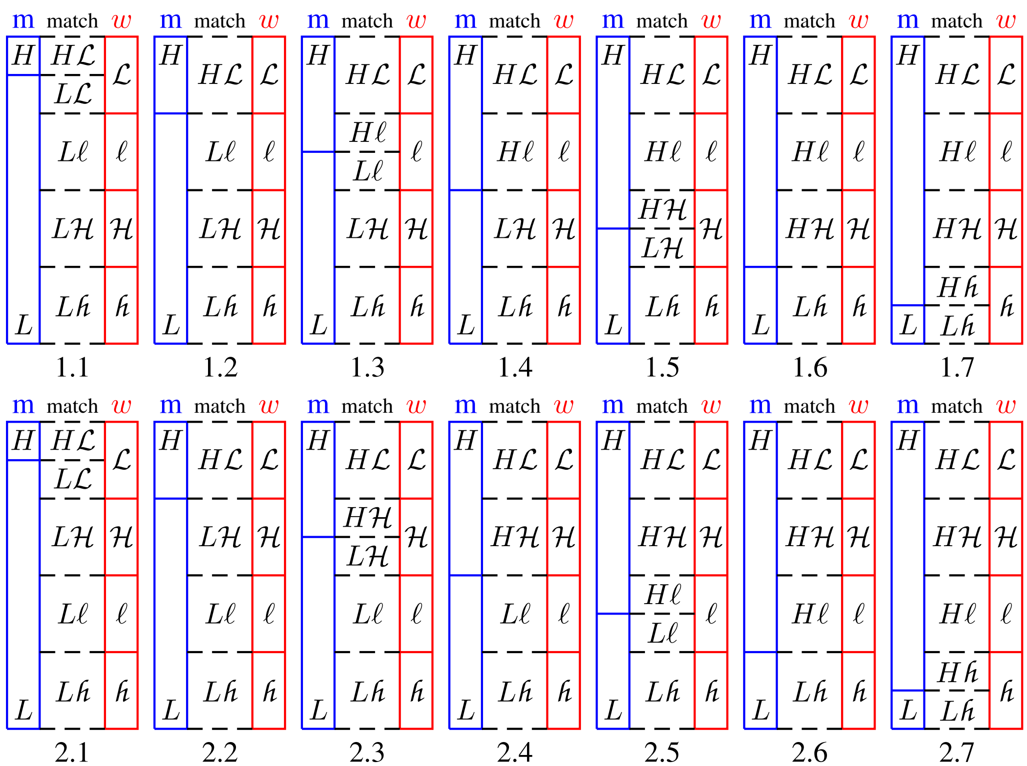

FIGURE 3. STABLE MATCHINGS UNDER INCOME-INCOME SUBMODULARITY AND INCOME-FITNESS SUPERMODULARITY.

Note: Type $\mathcal{L}$ women almost always marry higher-income husbands than other women do. Type $\hbar$ women almost always marry lower-income husbands than other women do. When $s_{H\ell} + s_{L\mathcal{H}} > s_{H\mathcal{H}} + s_{L\ell}$, a type ℓ woman almost always marries a weakly higher/lower-income husband than a type $\mathcal{H}$ woman in a stable matching (Markets 1.1-1.7). When $s_{H\ell} + s_{L\mathcal{H}} < s_{H\mathcal{H}} + s_{L\ell}$, a type ℓ woman almost always marries a weakly higher/lower-income husband than a type $\mathcal{H}$ woman in a stable matching (Markets 2.1-2.7).

TABLE 5—STABLE MARRIAGE PAYOFF DIFFERENCES UNDER INCOME-INCOME SUBMODULARITY AND INCOME-FITNESS SUPERMODULARITY.

	$v_{mH} - v_{mL}$		$v_{w\mathcal{H}} - v_{w\mathcal{L}}$
1.1	$s_{H\mathcal{L}} - s_{L\mathcal{L}}$	$=$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
1.2	$\lambda(s_{H\ell} - s_{L\ell}) + (1-\lambda)(s_{H\mathcal{L}} - s_{L\mathcal{L}})$	$=$	$\lambda(s_{L\mathcal{H}} - s_{L\ell} + s_{H\ell} - s_{H\mathcal{L}}) + (1-\lambda)(s_{L\mathcal{H}} - s_{L\mathcal{L}})$
1.3	$s_{H\ell} - s_{L\ell}$	$=$	$s_{L\mathcal{H}} - s_{L\ell} + s_{H\ell} - s_{H\mathcal{L}}$
1.4	$\lambda(s_{H\mathcal{H}} - s_{L\mathcal{H}}) + (1-\lambda)(s_{H\ell} - s_{L\ell})$	$=$	$\lambda(s_{H\mathcal{H}} - s_{H\mathcal{L}}) + (1-\lambda)(s_{L\mathcal{H}} - s_{L\ell} + s_{H\ell} - s_{H\mathcal{L}})$
1.5	$s_{H\mathcal{H}} - s_{L\mathcal{H}}$	$=$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$
1.6	$\lambda(s_{H\hbar} - s_{L\hbar}) + (1-\lambda)(s_{H\mathcal{H}} - s_{L\mathcal{H}})$	$\leq$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$
1.7	$s_{H\hbar} - s_{L\hbar}$	$<$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$
2.1	$s_{H\mathcal{L}} - s_{L\mathcal{L}}$	$=$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.2	$\lambda(s_{H\mathcal{H}} - s_{L\mathcal{H}}) + (1-\lambda)(s_{H\mathcal{L}} - s_{L\mathcal{L}})$	$=$	$\lambda(s_{H\mathcal{H}} - s_{H\mathcal{L}}) + (1-\lambda)(s_{L\mathcal{H}} - s_{L\mathcal{L}})$
2.3	$s_{H\mathcal{H}} - s_{L\mathcal{H}}$	$=$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$
2.4	$\lambda(s_{H\ell} - s_{L\ell}) + (1-\lambda)(s_{H\mathcal{H}} - s_{L\mathcal{H}})$	$\leq$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$
2.5	$s_{H\ell} - s_{L\ell}$	$<$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$
2.6	$\lambda(s_{H\hbar} - s_{L\hbar}) + (1-\lambda)(s_{H\ell} - s_{L\ell})$	$<$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$
2.7	$s_{H\hbar} - s_{L\hbar}$	$<$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$

TABLE 6—STABLE MARRIAGE PAYOFF DIFFERENCES UNDER INCOME-INCOME SUPERMODULARITY AND INCOME-FITNESS SUBMODULARITY.

	$v_{mH} - v_{mL}$		$v_{w\mathcal{H}} - v_{w\mathcal{L}}$
1.1	$s_{H\hbar} - s_{L\hbar}$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
1.2	$\lambda(s_{H\mathcal{H}} - s_{L\mathcal{H}}) + (1-\lambda)(s_{H\hbar} - s_{L\hbar})$	$>$	$\lambda(s_{L\mathcal{H}} - s_{L\mathcal{L}})$
1.3	$s_{H\mathcal{H}} - s_{L\mathcal{H}}$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
1.4	$\lambda(s_{H\ell} - s_{L\ell}) + (1-\lambda)(s_{H\mathcal{H}} - s_{L\mathcal{H}})$	?	$\lambda(s_{H\mathcal{H}} - s_{H\ell} + s_{L\ell} - s_{L\mathcal{L}}) + (1-\lambda)(s_{L\mathcal{H}} - s_{L\mathcal{L}})$
1.5	$s_{H\ell} - s_{L\ell}$	?	$s_{H\mathcal{H}} - s_{H\ell} + s_{L\ell} - s_{L\mathcal{L}}$
1.6	$\lambda(s_{H\mathcal{L}} - s_{L\mathcal{L}}) + (1-\lambda)(s_{H\ell} - s_{L\ell})$	?	$\lambda(s_{H\mathcal{H}} - s_{H\mathcal{L}}) + (1-\lambda)(s_{H\mathcal{H}} - s_{H\ell} + s_{L\ell} - s_{L\mathcal{L}})$
1.7	$s_{H\mathcal{L}} - s_{L\mathcal{L}}$	$<$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.1	$s_{H\hbar} - s_{L\hbar}$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.2	$\lambda(s_{H\ell} - s_{L\ell}) + (1-\lambda)(s_{H\hbar} - s_{L\hbar})$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.3	$s_{H\ell} - s_{L\ell}$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.4	$\lambda(s_{H\mathcal{H}} - s_{L\mathcal{H}}) + (1-\lambda)(s_{H\ell} - s_{L\ell})$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.5	$s_{H\mathcal{H}} - s_{L\mathcal{H}}$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.6	$\lambda(s_{H\mathcal{L}} - s_{L\mathcal{L}}) + (1-\lambda)(s_{H\mathcal{H}} - s_{L\mathcal{H}})$	?	$\lambda(s_{H\mathcal{H}} - s_{H\mathcal{L}}) + (1-\lambda)(s_{L\mathcal{H}} - s_{L\mathcal{L}})$
2.7	$s_{H\mathcal{L}} - s_{L\mathcal{L}}$	$<$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$

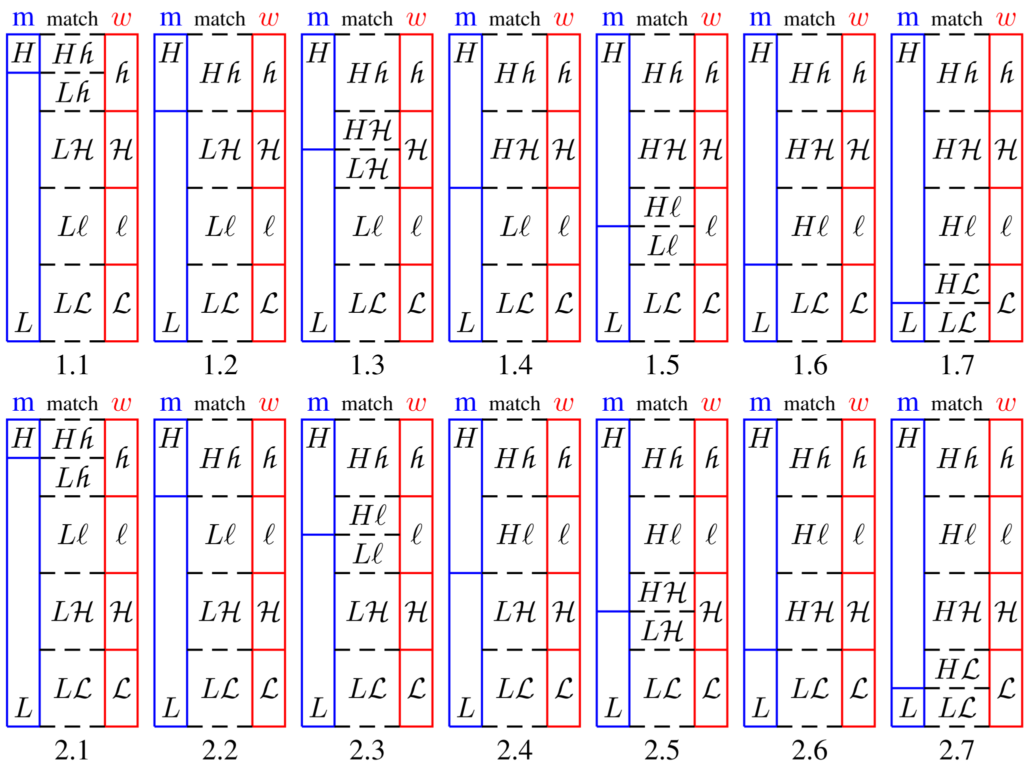

FIGURE 4. STABLE MATCHINGS UNDER INCOME-INCOME SUPERMODULARITY AND INCOME-FITNESS SUBMODULARITY.

Note: Type $\hbar$ women almost always marries higher-income husbands than other women do, and type $\mathcal{L}$ women almost always marry lower-income husbands than other women do. When $s_{H\mathcal{H}} + s_{L\ell} > s_{H\ell} + s_{L\mathcal{H}}$, a type $\mathcal{H}$ woman almost always marries a weakly higher-income husband than a type ℓ woman does (Markets 1.1-1.7). When $s_{H\mathcal{H}} + s_{L\ell} < s_{H\ell} + s_{L\mathcal{H}}$, a type $\mathcal{H}$ woman almost always marries a weakly lower-income husband than a type ℓ woman does (Markets 2.1-2.7).

TABLE 7—STABLE MARRIAGE PAYOFF DIFFERENCES UNDER INCOME-INCOME SUBMODULARITY AND INCOME-FITNESS SUBMODULARITY.

	$v_{mH} - v_{mL}$		$v_{w\mathcal{H}} - v_{w\mathcal{L}}$
1.1	$s_{H\ell} - s_{L\ell}$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
1.2	$\lambda(s_{H\mathcal{L}} - s_{L\mathcal{L}}) + (1-\lambda)(s_{H\ell} - s_{L\ell})$	$\geq$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
1.3	$s_{H\mathcal{L}} - s_{L\mathcal{L}}$	$=$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
1.4	$\lambda(s_{H\hbar} - s_{L\hbar}) + (1-\lambda)(s_{H\mathcal{L}} - s_{L\mathcal{L}})$	$=$	$\lambda(s_{L\mathcal{H}} - s_{L\hbar} + s_{H\hbar} - s_{H\mathcal{L}}) + (1-\lambda)(s_{L\mathcal{H}} - s_{L\mathcal{L}})$
1.5	$s_{H\hbar} - s_{L\hbar}$	$=$	$s_{L\mathcal{H}} - s_{L\hbar} + s_{H\hbar} - s_{H\mathcal{L}}$
1.6	$\lambda(s_{H\mathcal{H}} - s_{L\mathcal{H}}) + (1-\lambda)(s_{H\hbar} - s_{L\hbar})$	$=$	$\lambda(s_{H\mathcal{H}} - s_{H\mathcal{L}}) + (1-\lambda)(s_{L\mathcal{H}} - s_{L\hbar} + s_{H\hbar} - s_{H\mathcal{L}})$
1.7	$s_{H\mathcal{H}} - s_{L\mathcal{H}}$	$=$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$
2.1	$s_{H\ell} - s_{L\ell}$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.2	$\lambda(s_{H\hbar} - s_{L\hbar}) + (1-\lambda)(s_{H\ell} - s_{L\ell})$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.3	$s_{H\hbar} - s_{L\hbar}$	$>$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.4	$\lambda(s_{H\mathcal{L}} - s_{L\mathcal{L}}) + (1-\lambda)(s_{H\hbar} - s_{L\hbar})$	$\geq$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.5	$s_{H\mathcal{L}} - s_{L\mathcal{L}}$	$=$	$s_{L\mathcal{H}} - s_{L\mathcal{L}}$
2.6	$\lambda(s_{H\mathcal{H}} - s_{L\mathcal{H}}) + (1-\lambda)(s_{H\mathcal{L}} - s_{L\mathcal{L}})$	$=$	$\lambda(s_{H\mathcal{H}} - s_{H\mathcal{L}}) + (1-\lambda)(s_{L\mathcal{H}} - s_{L\mathcal{L}})$
2.7	$s_{H\mathcal{H}} - s_{L\mathcal{H}}$	$=$	$s_{H\mathcal{H}} - s_{H\mathcal{L}}$

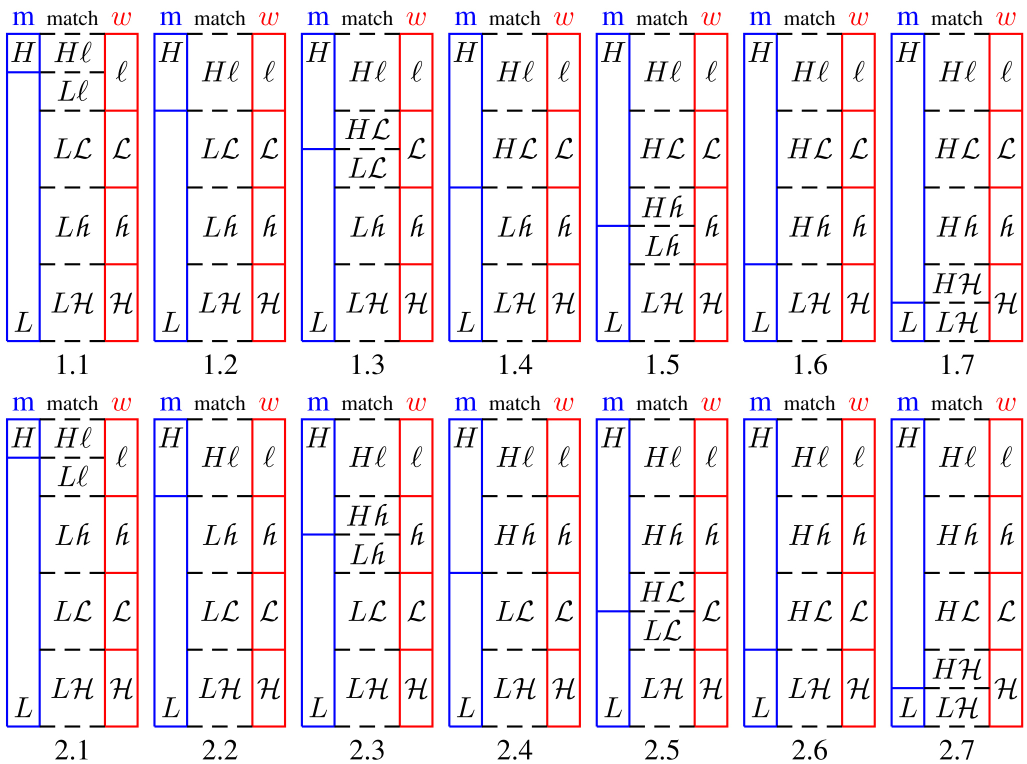

FIGURE 5. STABLE MATCHINGS UNDER INCOME-INCOME SUBMODULARITY AND INCOME-FITNESS SUBMODULARITY.

Note: Type ℓ women almost always marry higher-income husbands than other women do. Type ℓ women almost always marry lower-income husbands than other women do. When $s_{H\mathcal{L}} + s_{L\hbar} > s_{H\hbar} + s_{L\mathcal{L}}$, a type $\mathcal{L}$ woman almost always marries a weakly higher/lower-income husband than a type h woman in a stable matching (Markets 1.1-1.7). When $s_{H\mathcal{L}} + s_{L\hbar} < s_{H\hbar} + s_{L\mathcal{L}}$, a type $\mathcal{L}$ woman almost always marries a weakly higher/lower-income husband than a type h woman in a stable matching (Markets 2.1-2.7).

$p_m = p_w$, $y_{m\mathrm{H}} - y_{m\mathrm{L}} = y_{w\mathrm{H}} - y_{w\mathrm{L}}$, $s_{H\mathcal{L}} = s_{L\mathcal{H}}$, and $r = 1$), the same number of men and women go to college.

PROOF:

Since the equilibrium college enrollment rate is

$$F_w(c^*_{w1}) = F_w[p_w(y_{w\mathrm{H}} - y_{w\mathrm{L}}) + p_w(v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}})],$$

it suffices to show that $v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}}$ decreases as r increases. I follow the notation in the proof of Theorem 1. Since $(v_{w\mathcal{H}} - v_{w\mathcal{L}})_x$ increases in x, it suffices to show that x^* decreases in r. As r increases, $G_{mH}(x^*)$ does not change and $G_{w\mathcal{H}}(x^*)$ and $G_{w\mathcal{H}}(x^*) + G_{w\hbar}(x^*)$ increase, because

$$\begin{aligned} \frac{dG_{w\mathcal{H}}}{dr} &= \frac{d[F(c^*_{w2})(1-p_w)p_w r]}{dr} = (1-p_w)p_w\left[F(c^*_{w2}) + f(c^*_{w2})\frac{dc^*_{w2}}{dr}\right], \\ \frac{d(G_{w\mathcal{H}} + G_{w\hbar})}{dr} &= \frac{d[F(c^*_{w2})(1-p_w)p_w]}{dr} = f(c^*_{w2})(1-p_w)p_w\frac{dc^*_{w2}}{dr}. \end{aligned}$$

Therefore, $\phi_1(x)$ and $\phi_2(x)$ decrease in r, and as a result, x^* weakly decreases. If the setting is completely gender-symmetric with $r = 1$, then $v^*_{mH} - v^*_{mL} = v^*_{w\mathcal{H}} - v^*_{w\mathcal{L}} = \frac{1}{2}(s_{H\mathcal{H}} - s_{L\mathcal{L}})$. In equilibrium, all college women invest in career and the same number of men and women go to college.

QED.

VI. Investing and Marrying in the Same Period is Weakly Dominated

I prove the claim that investing and entering the marriage market in the same period is weakly dominated by investing and then entering the marriage market when the investment return is realized. In the model in the main text, investing in a period has precluded an agent from entering the marriage market in the same period. I allow now men and women to enter the marriage market at the same time as they invest. Because the income uncertainty associated with the investment has not been resolved when they enter the marriage market, their marriage characteristics cannot be simply represented by realized incomes, but rather by distributions of incomes; for women, also possibly by distributions of fitnesses. The model needs to be expanded to include such marriage characteristics. See Borch (1962); Wilson (1968); Chiappori and Reny (2016).

Suppose that men and women can enter the marriage market when they make a college or career investment. Their feasible strategy is not simply to invest ($I_a, a = 1, 2$) or not to invest ($N_a, a = 1, 2$), but also to delay marriage ($D_a, a = 1, 2$) or to enter the marriage market immediately ($E_a, a = 1, 2$). Assume that their investment strategies are publicly known and they can commit to the investments. An age 1 man can choose among four actions: going to college and delaying marriage (I_1, D_1), to going to college and entering the marriage market in the current period (I_1, E_1), not going to college and entering the marriage market immediately (N_1, E_1), and not going to college and delaying marriage (N_1, D_1). The last option of not going to college and delaying marriage

(N_1, D_1) is never chosen because there is nothing to be gained when a man's characteristic is certain. An age 2 man who has failed the initial college investment also chooses among four actions: investing in career and delaying marriage (I_2, D_2), investing in career and entering the marriage market in the current period (I_2, E_2), not investing in career and entering the marriage market (N_2, E_2), and not going to college and delaying entrance to marriage market (N_2, D_2). Again, (N_2, D_2) is weakly dominated by (N_2, E_2). Women's expanded strategies are similarly represented.

If agents choose to enter the marriage market early, their income is not realized. A man's marriage type may not be represented by an income y_m. It could be a distribution. Let P_m represent a man who has a realized income from distribution P_m in the next period. Similarly, a woman's marriage type is her income and fitness distribution $\equiv P_w$. The expected marriage surplus of a type P_m man and a type $\mathcal{H}$ woman is $\widetilde{s}_{P_m\mathcal{H}} = \int s_{\tau_m\mathcal{H}} dP_m$. For example, a college-investing type $p_m \circ H + (1-p_m) \circ L$ man and a type $\mathcal{H}$ woman generate surplus $p_m s_{H\mathcal{H}} + (1-p_m) s_{H\mathcal{L}}$. Let T_m and T_w represent expanded sets of men and women's marriage types, $\widetilde{G}_m$ and $\widetilde{G}_w$ measures of the marriage types, $\widetilde{G}$ the matching measure on the marriage-types. Let $\widetilde{v}_m$ and $\widetilde{v}_w$ be men and women's marriage payoff functions. A marriage market's outcome $(\widetilde{G}, \widetilde{v}_m, \widetilde{v}_w)$ is stable if $\widetilde{G}$ has a marginals $\widetilde{G}_m$ and $\widetilde{G}_w$, $\widetilde{v}_{m\tau_m} \geq 0$, $\widetilde{v}_{w\tau_w} \geq 0$ for all $\tau_m \in T_m$ and $\tau_w \in T_w$, $\widetilde{v}_{m\tau_m} + \widetilde{v}_{w\tau_w} = \widetilde{s}_{\tau_m\tau_w}$ when $\widetilde{G}(\tau_m, \tau_w) > 0$, and $\widetilde{v}_{m\tau_m} + \widetilde{v}_{w\tau_w} \geq \widetilde{s}_{\tau_m\tau_w}$ for all τ_m, τ_w in the support of $\widetilde{G}_m$ and $\widetilde{G}_w$.

Men Investing and Marrying in Period 2. First, consider a low-income man who has failed the initial college investment and is deciding between to invest in career and delay marriage (I_2, D_2) and to invest in career and enter the marriage market immediately (I_2, E_2). The two actions give him the same expected income and incur the same investment cost. The only difference is in the marriage payoffs. His expected marriage payoff from (I_2, D_2) is $p_m\widetilde{v}_{mH} + (1-p_m)\widetilde{v}_{mL}$. His expected marriage payoff from (I_2, E_2) is $\widetilde{v}_{mP_m}$, where $P_m = p_m \circ H + (1-p_m) \circ L$.

Suppose that a P_m man marries a τ_w woman in the stable matching, and gets the stable marriage payoff

$$\widetilde{v}_{mP_m} = \widetilde{s}_{P_m\tau_w} - \widetilde{v}_{w\tau_w} = p_m\widetilde{s}_{H\tau_w} + (1-p_m)\widetilde{s}_{L\tau_w} - \widetilde{v}_{w\tau_w}.$$

By the no blocking pair condition, the marriage payoff of each type of man weakly exceeds what he would get if he marries a type τ_w woman, in particular, $\widetilde{v}_{mH} \geq \widetilde{s}_{H\tau_w} - \widetilde{v}_{w\tau_w}$ and $\widetilde{v}_{mL} \geq \widetilde{s}_{L\tau_w} - \widetilde{v}_{w\tau_w}$. Adding the two inequalities,

$$p_m\widetilde{v}_{mH} + (1-p_m)\widetilde{v}_{mL} \geq p_m\widetilde{s}_{H\tau_w} + (1-p_m)\widetilde{s}_{L\tau_w} - \widetilde{v}_{w\tau_w} = \widetilde{v}_{mP_m}.$$

If one of the two inequalities is strict, then investing and entering the marriage market at the same time is strictly dominated.

Men Investing and Marrying in Period 1. Next, consider a man who decides between to go to college and delay marriage (I_1, D_1) and to go to college and enter the marriage market in the same period (I_1, E_1). The man will make a career investment in the subsequent period if necessary. His total expected payoff from (I_1, D_1) is

$$p_m(y_{m\mathrm{H}} + \widetilde{v}_{mH}) + (1 - p_m)(y_{m\mathrm{L}} + \widetilde{v}_{mL}) + \max\{0, p_m(y_{m\mathrm{H}} - y_{m\mathrm{L}} + \widetilde{v}_{mH} - \widetilde{v}_{mL}) - c_m\}.$$

Let $\delta_m \in [0, 1]$ be the probability that the man makes a career investment after a failed college investment. The total expected payoff becomes simply

$$p_m(y_{m\mathrm{H}} + \widetilde{v}_{mH}) + (1 - p_m)(y_{m\mathrm{L}} + \widetilde{v}_{mL}) + \delta_m[p_m(y_{m\mathrm{H}} - y_{m\mathrm{L}} + \widetilde{v}_{mH} - \widetilde{v}_{mL}) - c_m].$$

His total expected payoff from (I_1, E_1) while following the same probability δ_m of career investment is

$$\widetilde{v}_{mP_m} + p_m y_{m\mathrm{H}} + (1 - p_m)y_{m\mathrm{L}} + \delta_m[p_m(y_{m\mathrm{H}} - y_{m\mathrm{L}}) - c_m],$$

where $P_m = [p_m + (1 - p_m)\delta_m p_m] \circ H + [(1 - p_m)(1 - \delta_m p_m)] \circ L$. The expected income is the same with the two strategies. The only difference is in expected marriage payoff. The difference is

$$[p_m + (1 - p_m)\delta_m p_m]\widetilde{v}_{mH} + [(1 - p_m)(1 - \delta_m p_m)]\widetilde{v}_{mL} - \widetilde{v}_{mP_m}.$$

We have shown in the previous argument that

$$p_m\widetilde{v}_{mH} + (1 - p_m)\widetilde{v}_{mL} \geq \widetilde{v}_{mP_m},$$

where $P_m = p_m \circ H + (1 - p_m) \circ L$. The argument does not depend on the particular choice of p_m, so we have

$$[p_m + (1 - p_m)\delta_m p_m]\widetilde{v}_{mH} + [(1 - p_m)(1 - \delta_m p_m)]\widetilde{v}_{mL} - \widetilde{v}_{mP_m} \geq 0,$$

where $P_m = [p_m + (1 - p_m)\delta_m p_m] \circ H + [(1 - p_m)(1 - \delta_m p_m)] \circ L$. Hence, (I_1, E_1) followed by any choice of career investment is weakly dominated by (I_1, D_1) followed by the same choice of career investment.

Women Investing and Marrying in Period 2. Similarly, consider a low-income fit woman who decides between investing in career and delaying marriage (I_2, D_2) and investing in career and entering the marriage market in the same period (I_2, E_2). Assume that her income and fitness uncertainty is realized after she finishes her investment. Either action gives her the same expected income. If she invests and marries later (I_2, D_2), her expected marriage payoff is

$$p_w r\widetilde{v}_{w\mathcal{H}} + p_w(1 - r)\widetilde{v}_{w\hbar} + (1 - p_w)r\widetilde{v}_{w\mathcal{L}} + (1 - p_w)(1 - r)\widetilde{v}_{w\ell}.$$

If she invests and enters the marriage market immediately (I_2, E_2), her expected payoff is $\widetilde{v}_{wP_w}$ where $P_w = p_w r \circ \mathcal{H} + p_w(1 - r) \circ \hbar + (1 - p_w)r \circ \mathcal{L} + (1 - p_w)(1 - r) \circ \ell$. Suppose that a P_w woman marries a type τ_m man, and they divide up their surplus:

$\widetilde{v}_{wP_w} = \widetilde{s}_{\tau_m P_w} - \widetilde{v}_{m\tau_m}$, where

$$\widetilde{s}_{\tau_m P_w} = p_w r \widetilde{s}_{\tau_m \mathcal{H}} + p_w(1-r)\widetilde{s}_{\tau_m \hbar} + (1-p_w) r \widetilde{s}_{\tau_m \mathcal{L}} + (1-p_w)(1-r)\widetilde{s}_{\tau_m \ell}.$$

By the no blocking pair condition, for every type $\tau_w \in \{\mathcal{H}, \mathcal{L}, \hbar, \ell\}$, $\widetilde{v}_{w\tau_w} \geq \widetilde{s}_{\tau_m \tau_w} - \widetilde{v}_{m\tau_m}$. Hence,

$$\begin{aligned} & p_w r \widetilde{v}_{w\mathcal{H}} + p_w(1-r)\widetilde{v}_{w\hbar} + (1-p_w) r \widetilde{v}_{w\mathcal{L}} + (1-p_w)(1-r)\widetilde{v}_{w\ell} \\ \geq \;& \left[p_w r \widetilde{s}_{\tau_m \mathcal{H}} + p_w(1-r)\widetilde{s}_{\tau_m \hbar} + (1-p_w) r \widetilde{s}_{\tau_m \mathcal{L}} + (1-p_w)(1-r)\widetilde{s}_{\tau_m \ell}\right] - \widetilde{v}_{m\tau_m} \\ = \;& \widetilde{s}_{\tau_m P_w} - \widetilde{v}_{m\tau_m} = \widetilde{v}_{wP_w}. \end{aligned}$$

Therefore, (I_2, E_2) is weakly dominated by (I_2, D_2). In fact, following the argument above, for any $p_{w\mathcal{H}} \geq 0$, $p_{w\hbar} \geq 0$, and $p_{w\mathcal{L}} \geq 0$,

$$p_{w\mathcal{H}}\widetilde{v}_{w\mathcal{H}} + p_{w\hbar}\widetilde{v}_{w\hbar} + p_{w\mathcal{L}}\widetilde{v}_{w\mathcal{L}} + (1 - p_{w\mathcal{H}} - p_{w\hbar} - p_{w\mathcal{L}})\widetilde{v}_{w\ell} \geq \widetilde{v}_{wP_w},$$

where

$$P_w = p_{w\mathcal{H}} \circ \mathcal{H} + p_{w\hbar} \circ \hbar + p_{w\mathcal{L}} \circ \mathcal{L} + (1 - p_{w\mathcal{H}} - p_{w\hbar} - p_{w\mathcal{L}}) \circ \ell.$$

Women Investing and Marrying in Period 1. Consider a woman who decides between going to college and delaying marriage (I_1, D_1) and going to college and entering the marriage market (I_1, E_1), followed by a known δ_w probability of career investment if the college investment fails. Her expected marriage payoff of (I_1, D_1) followed by probability δ_w of (I_2, D_2) and probability $(1-\delta_w)$ of (N_2, E_2) in case of college investment failure is

$$\begin{aligned} & [p_w + (1-p_w)p_w r \delta_w]\widetilde{v}_{w\mathcal{H}} + (1-p_w)[(1-\delta_w) + \delta_w(1-p_w)r]\widetilde{v}_{w\mathcal{L}} \\ & + (1-p_w)\delta_w p_w (1-r)\widetilde{v}_{w\hbar} + (1-p_w)\delta_w(1-p_w)(1-r)\widetilde{v}_{w\ell}. \end{aligned}$$

Her expected marriage payoff of (I_1, E_1) followed by probability δ_w of (I_2, D_2) and probability $(1-\delta_w)$ of (N_2, E_2) in case of college investment failure is $\widetilde{v}_{wP_w}$ where

$$\begin{aligned} P_w \;=\; & [p_w + (1-p_w)p_w r \delta_w] \circ \mathcal{H} + (1-p_w)[(1-\delta_w) + \delta_w(1-p_w)r] \circ \mathcal{L} \\ & + (1-p_w)\delta_w p_w(1-r) \circ \hbar + (1-p_w)\delta_w(1-p_w)(1-r) \circ \ell. \end{aligned}$$

By the same argument as in "Women Investing and Marrying in Period 1,"

$$\begin{aligned} & [p_w + (1-p_w)p_w r \delta_w]\widetilde{v}_{w\mathcal{H}} + (1-p_w)[(1-\delta_w) + \delta_w(1-p_w)r]\widetilde{v}_{w\mathcal{L}} \\ & \quad + (1-p_w)\delta_w p_w(1-r)\widetilde{v}_{w\hbar} + (1-p_w)\delta_w(1-p_w)(1-r)\widetilde{v}_{w\ell} \;\geq\; \widetilde{v}_{wP_w}. \end{aligned}$$

Hence, investing and marrying simultaneously in period 1 is weakly dominated by investing and waiting to marry after the investment outcome is realized.

Empirical evidence also supports that agents tend to marry after they have finished their investments. In 1979 NLSY, overwhelming majority, 79% of women and 84% of men finished their schooling before marriage (Browning, Chiappori and Weiss, 2014). Furthermore, Oppenheimer (1988) makes the similar argument from a sociological point of view that people only marry after they have clear future job prospects. On the flip side, however, not everyone waits until he or she finishes all the schooling to marry. A possible reason for early marriage is that the agents enjoy flow consumption of marriages while their uncertainty about future income is low.

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