Two-Player Zero-Sum Poker Models with One and Two Rounds of Betting

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Abstract

The paper presents the basic von Neumann's two-player zero-sum poker model with independent and identically distributed uniform hands, and extends it by allowing player II to re-raise. The analysis shows that both models favor the player who initially raises, but re-raising option cuts in player I's advantage, reducing his expected payoff. "Payoff square", a square diagram that indicates the payoffs under different hands and strategies, is introduced and used to derive players' payoffs. Extensions to multiple and infinite rounds of betting are discussed, and optimal strategies are conjectured. Related models are reviewed along the way.

1 Introduction

Poker is a complex multi-player game of chance and deception. In order to gain insight into different aspects of the game, mathematical and psychological alike, we construct simple models of poker by making assumptions about their hands and restricting rules. Two-player zero-sum poker models with independent uniform hands are the simplest non-trivial ones. von Neumann (1953) discusses his model in *Theory of Games and Economic Behaviors*[von Neumann & Morgenstern, 1953].

In this paper, in addition to presentation of the original model, it is extended to allow an additional round of raise by the second player. This modification makes the model closer and more applicable to real poker (Texas Hold'em), and provides more information on optimal players by both players in a more complicated situation. In addition, discussion on allowing multiple rounds follows.

2 von Neumann's Model

This section presents the basic model of von Neumann, and definitions and lemmas that are applicable in general. Two players each contribute an ante of $1, and are dealt

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"hands" x_1 and x_2, respectively independent and identically distributed as $U(0,1)$. Player I can check or bet a predetermined amount B, and player II can call or fold if player I bets. The only available information for each player is his own hand and the game structure. Strategies for two players are

Player I: $s_1 : x_1 \to \{\text{check, bet}\}$;

Player II: $s_2 : x_2 \times s_1(x_1) \to \{\text{call, fold}\}$.

A player's payoff is dependent of $x_1, x_2, s_1(x_1), s_2(x_2)$. Extensive form of the game and optimal strategies are presented in Figure 1 with player I's payoffs shown (their reciprocals are player II's because the game is zero-sum).

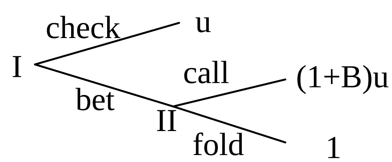

Figure 1: Extensive form of von Neumann's game, where $u = 1$ if $x_1 \geq x_2; = -1$ otherwise.

We want to investigate how players play in equilibrium. They are going to play a pair of optimal strategies as defined below. A strategy is optimal if given any hand and the other player's strategy, there is no incentive to deviate to any other strategy.

Definition 2.1. *For player i, a strategy s_i^* is* **optimal** *if given any other strategy s_i' and other player's strategy s_j,*

$$\int_{x_j} u_i(x_i, x_j, s_i^*(x_i), s_j(x_j))dx_j \geq \int_{x_j} u_i(x_i, x_j, s_i'(x_i), s_j(x_j))dx_j \quad \forall x_1 \in (0,1).$$

For two similar hands, payoff from an optimal strategy should be the same. Otherwise, there is an incentive to deviate to the strategy played if given the other hand. Therefore, hands slightly bigger and slightly smaller yield similar payoffs. This idea is embodied in the **indifference condition (IC)**. For example in the optimal strategy above, player II is indifferent between folding and calling when he is dealt c.

Lemma 2.2 (IC)**.** *For s_i^*, as $\epsilon \to 0^+$,*

$$\int_{x_j} u_i\big(x_i, x_j, s_i^*\big(x_i - \epsilon\big), x_j\big) \to \int_{x_j} u_i\big(x_i, x_j, s_i^*\big(x_i + \epsilon\big), x_j\big) \quad \forall x_i. \tag{2.1}$$

Theorem 2.3. *An equilibrium strategy of von Neumann's game is presented as follows. Player I checks when $a \leq x_1 \leq b$ and bets amount B otherwise; in case of raise, player II calls if $x_2 > c$ and folds otherwise, where*

$$a = \frac{B}{(B+1)(B+4)}, \quad b = \frac{B^2 + 4B + 2}{(B+1)(B+4)}, \quad c = \frac{B(B+3)}{(B+1)(B+4)}.$$

Two players' optimal strategies are illustrated in Figure 2.

Proof. Optimal strategies are found by backward induction. Player II's optimal strategy is found first. When Player I raises, player II calls if his expected payoff is greater than -1, which is his payoff from folding. Since his payoff depends piecewise-linearly

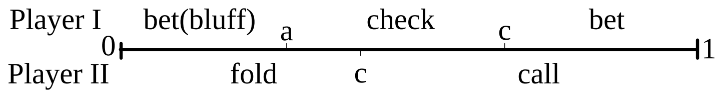

Figure 2: Optimal strategies of both players

on his hand strength x_2, $u_2(x_1, x_2)$, player II's payoff, is a monotonic function of x_2 if he calls. Therefore, player II's optimal strategy is to call when $x_2 > c$ and to fold otherwise.

Given player II's optimal strategy, player I should bet if his expected payoff of betting is greater than of checking (Tie situations need not be considered because the density function is non-atomic). Player I's payoff given x_1

- from checking is $(+1) \cdot (x_1 - 0) + (-1) \cdot (1 - x_1) = 2x_1 - 1$.
- from betting is

$$\int_0^c (+1)dx_2 + \int_c^1 (-1-B)dx_2 = c + (-1-B)(1-c) = (B+2)c - B - 1, \quad x_1 < c,$$

or

$$\int_0^c (+1)dx_2 + \int_c^{x_1} (1+B)dx_2 + \int_{x_1}^1 (-1-B)dx_2 = (B+1)(2x_1-1) - Bc, \quad x_1 \geq c.$$

Then by IC in Equation 2.1,

$$2x_1 - 1 = 2c + Bc - B - 1, \tag{2.2}$$
$$2x_1 - 1 = (B+1)(2x_1 - c - 1) + c \tag{2.3}$$

Since I's payoff is also piecewise linear with respect to x_1, player I's optimal strategy is to bet if $x_1 < a$ or $x_1 > b$, and to check if $a \leq x_1 \leq b$. Then $x_1 \leq c$ in Equation 2.2, and $x_1 \geq c$ in Equation 2.3,

$$2a - 1 = 2c + Bc - B - 1 \tag{2.4}$$
$$2b - 1 = (B+1)(2b - c - 1) + c \tag{2.5}$$

Player II's optimal strategy should obey the indifference condition,

$$\Big[a(1+B) + (1-b)(-1-B)\Big] \Big/ \Big(a + 1 - b\Big) = -1 \tag{2.6}$$

Solve Equations 2.4 and 2.5 give values a and b and substitute into Equation 2.6, $c = B(B+3)/\big[(B+1)(B+4)\big]$. a and b as functions of B are obtained by re-substitution. □

Payoffs of both players can be determined. In addition, given that player I determines his bet amount before hands are assigned, the optimal B that maximizes the expected payoff is of interest. First, **payoff squares** that describe strategies and corresponding payoffs of players are introduced.

Definition 2.4. *A* **payoff square** *is a two-dimensional square diagram with hands of player I in x-axis, and player II's in y-axis. A point in the square indicates a hand pair* (x_1, x_2)*, and the payoff,* $u_1(x_1, x_2, s_1(x_1, x_2), s_2(x_1, x_2))$*, indicated at the point is resulted from the strategies played corresponding to the hands.*

Remark 2.5. *Payoffs of any strategy set can be depicted by the payoff square. It can also be generalized to n-dimension, which is equivalent to taking multiple integrals of n variables (Besides the payoff square, only a payoff "cube" depicting a three-player game is beneficial).*

Corollary 2.6. *Given that both players follow the optimal strategies described in Theorem 2.3, player I's payoff is $u_1(x_1, x_2, s_1, s_2) = a$. Optimal bet amount is $B^* = 2$.*

Proof. Expected payoff from player I checking all hands is 0, +1 below $x_1 = x_2$ and -1 above $x_1 = x_2$. Equivalently, the payoff differential from this strategy is illustrated and expected payoff would be the same overall, so we add 1 above $x_1 = x_2$, subtract 1, and cancel out a square region on the top of $+B$ and $-B$ to get the resulting square on the right. The original, complete payoff square as well as its geometric and algebraic manipulation are shown in Figure 3.

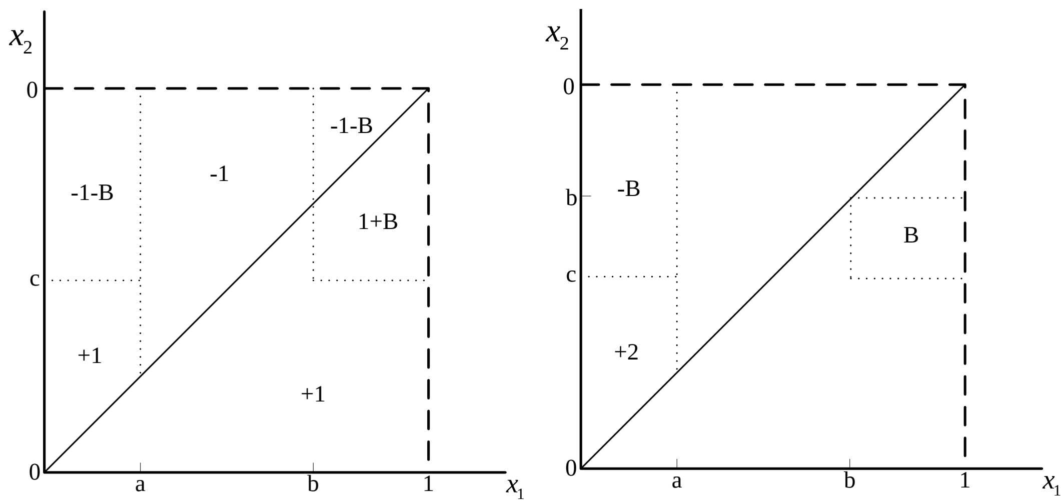

Figure 3: Illustration of Player I's payoff by payoff square.

Expected payoff of player I is

$$u_1 = B(1-b)(b-c) - B(a)(1-c) + (2)(1/2)(c+c-a)(a) = B/\big[(B+1)(B+4)\big] = a.$$

Maximizing $u_1(B)$ with respect to B, $\big(4 - B^2\big)/\big[(B+1)^2(B+4)^2\big] = 0, B^* = 2$. $u_1(2) = 1/9, a = 1/9, b = 7/9, c = 5/9$. □

The result deserves some discussion. Player I's payoff, $B/(B^2 + 5B + 4)$ is positive for all $B > 0$, and it achieves its maximum at $B^* = 2$, the pot size. This means that the game favors player I who is given the chance to raise, and he maximizes his payoff to be 1/9 by betting pot size every time. Player I has an advantage because he can bluff with his worst hands. More importantly, for real poker perhaps, he must bluff with his *worst* but not mediocre hands.

In contrast to von Neumann's model in which the bet amount is pre-determined and fixed, Donald Newman presents a model that has the same game structure but allows any bet amount [Newman, 1959] . Set $\xi = 2/(B+2)$. The optimal strategy is that I checks when $1/7 \le x_1 \le 4/7$, bets B with hands $(1 - 3\xi^2 + 2\xi^3)/7$, or $1 - 3\xi^2/7$; II calls if and only if $x_2 > 1 - 6\xi/7$. In this game, player I's value is 1/7 because player I bluffs 1/7 of time. Optimality of the strategy is proven by showing that the given pure strategy is a saddle point of all strategies.

3 Extension: Re-raise by player II

The key extension to the previous model is allowing player II to re-raise B_2 after calling player I's bet B_1, and I either calls or folds. This model is discussed in Ferguson's works [Ferguson & Ferguson, 2003], [Ferguson et al., 2007]. The strategies given hands $x_1, x_2 \sim U(0,1)$, are

Player I: $s_1 : x_1 \times s_2(s_1) \to \{\text{check, bet } B_1\} \times \{\text{fold, call}\} = \{\text{bet-fold, bet-raise, check}\}$,

Player II: $s_2 : x_2 \times$ {check, bet} $\to$ {bet B_2, check, fold}.

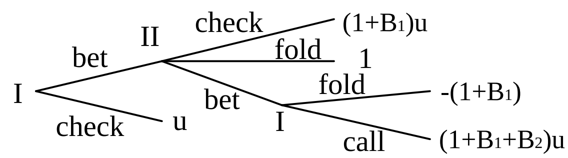

Figure 4: Extensive form of the two players with player I's payoffs.

Theorem 3.1. *Player I bet-folds when $x_1 \in (0,a] \cup (b,c]$, checks when $x_1 \in (a,b]$, and bet-calls when $x_1 > c$. Player II folds when $x_2 \le d$, calls when $x_2 \in (e,f]$, and re-raises when $x_1 \in (d,e] \cup (f,1]$. $0 \le a \le e \le b \le c \le f \le 1$ and $0 \le d \le e$ (d can be larger or smaller than a), where*

$$
\begin{aligned}
a &= \frac{B_1^2(2+2B_1+B_2)^2}{(1+B_1)\Delta}, b = 1 - \frac{2+B_1}{B_1}a, c = 1 - \frac{2B_1(2+B_1)(2+2B_1+B_2)}{\Delta}, \\
d &= \frac{B_1+2a}{2+B_1}, e = \frac{B_1}{1+B_1} - a, f = 1 - \frac{B_1(2+B_1)(2+2B_1+B_2)}{\Delta}
\end{aligned}
$$

where $\Delta = B_1(4+B_1)(2+2B_1+B_2)^2 + (1+B_1)(2+B_1)^2 B_2$. The optimal strategies are illustrated in Figure 5.

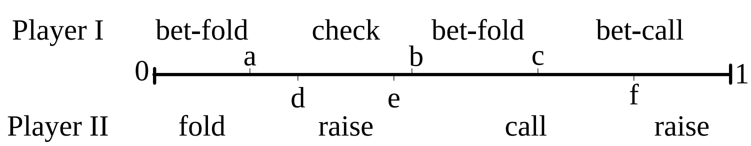

Figure 5: Optimal Strategies of both players.

Proof. First, apply the ICs for

- I at a: $-2a + (-2-B_1)d = B_1$;
- I at b: $2B_1 b + (2+B_1)d + (-2B_1-2)e = B_1$;
- I at c: $(-2-2B_1-B_2)d + (2+2B_1+B_2)e + B_2 f = B_2$;
- II at d: $(2+B_1)a - (2+B_1)b + (2+2B_1+B_2)c = B_1 + B_2$;
- II at e: $(-2-2B_1)b + (2+2B_1+B_2)c = B_2$;
- II at f: $-c + 2f = 1$.

Solving six linearly independent equations of six unknowns, we get results as presented in the theorem.

Now suppose we ignore the antes contributed by the players by treating them as sunk costs. Given that Player II uses the conjectured optimal strategy, and player I has hand x_1, I's "gain"

- from checking is $2x_1$.
- from bet-folding is $2a$ if $0 < x < e$; $2a + 2(1 + B)(x - e)$ if $e < x < f$, and $2a + 2(1 + B_1)(f - e)$ if $f < x < 1$.
- from bet-calling is $2a - 2(1+B_1+B_2)(e-d)$ if $0 < x < d$; $2a - 2(1+B_1+B_2)(e-x)$ if $d < x < e$; $2a + 2(1 + B_1)(x - e)$ if $e < x < f$; and $2a + 2(1 + B_1)(f - e) + 2(1 + B_1 + B_2)(x - f)$ if $f < x < 1$.

We can verify that at every critical point, the payoffs from two strategies are equal. Then noting that the payoff function is piecewise linear, we verify that player I's strategy is optimal. Similarly given player I's optimal strategy and Player II's hand x_2, Player II's expected payoff from

- folding is 0 if $0 < y < a$, $2(y - a)$ if $a < y < b$, and $2(b - a)$ if $b < y < 1$.
- calling is $-2(1 + B_1)(a - y)$ if $0 < y < a$; $2(y - a)$ if $a < y < b$; and $2(b - a) + 2(1 + B_1)(y - b)$ if $b < y < 1$.
- raising is 0 if $0 < y < a$, $2(y - a)$ if $a < y < b$; $2(b - a)$ if $b < y < c$; and $2(b - a) + (1 + B_1 + B_2)(y - c)$ if $c < y < 1$.

Then with II's boundary conditions verified, player II is proven to play an optimal strategy as well. Complicated operations were done by Maple 11. □

Corollary 3.2. *Expected payoff of player I is a. Optimal bet for player I,* $B_1^* = 1 + \sqrt{13}/3$. *Optimal bet for II is* $B_2^* = 2B_1^* + 2 = 4 + 2\sqrt{13}/3$.

Proof. Applying the modified payoff square as illustrated in Figure 6,

$$\begin{aligned} u_1 &= B_1\big[-a(1-d) - (c-b)(e-d) + (1-c)(e-d) + (1-b)(b-e)\big] + \\ &\quad 2\big[(2d-a)(a/2) - (c-b)(e-d)\big] + B_2\big[(1-c)(e-d) - (f-c)(1-f)\big] \\ &= B_1^2\Big/\{(1+B_1)\big[B_1(4+B_1) + (1+B_1)(2+B_1)^2 B_2/(2+2B_1+B_2)^2\big]\} = a. \end{aligned}$$

Minimizing u_1 with respect to B_2 while B_1 is fixed, is equivalent to maximizing $B_2/(2 + 2B_1 + B_2)^2$,

$$\frac{(2B_1 + 2 + B_2) - 2B_2}{(2 + 2B_1 + B_2)^2} = 0 \Rightarrow B_2^* = 2B_1^* + 2.$$

Substitute in, $u_1 = 8B_1^2\Big/\Big[(1 + B_1)(9B_1^2 + 36B_1 + 4)\Big]$. Then maximizing u_1 with respect to B_1 yields $9B_1^3 - 40B_1 - 8 = 0$. Three roots are $B_1 = -2, 1+\sqrt{13}/3, 1-\sqrt{13}/3$. Clearly,

$$B_1^* = 1 + \sqrt{13}/3 \approx 2.202, \quad B_2^* = 4 + 2\sqrt{13}/3 \approx 6.404$$

Substituting in B_1^* and B_2^*, $a = 0.0955, b = 0.8178, c = 0.909, d = 0.569, e = 0.592, f = 0.954$.

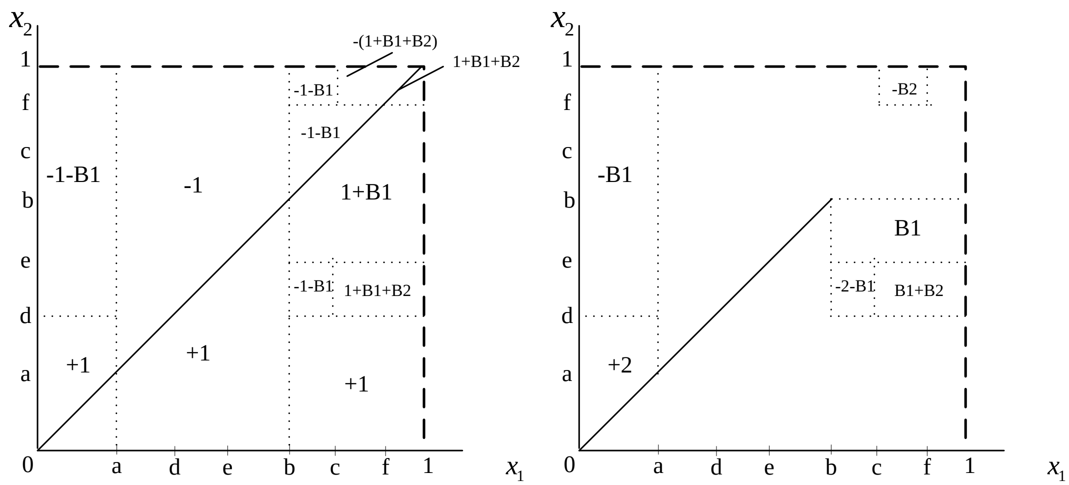

Figure 6: Payoff of Player I by payoff square.

Though the game still favors player I, its payoff (0.0955) is lower than that in the original von Neumann model (1/9). This shows that allowing player II to re-raise restricts player I's audacity of bluffing, thus decreasing his advantage and profit. Obviously player I always has advantage given that he makes a voluntary decision first because he can always checks to yield an expected payoff of 0. Also note that player I's optimal bet is a little over the pot size, and player II is little under the added pot size, however very close. This gives some insight and justification to bet by pot size in real poker.

In addition, player II can bluff with his "good" hands. This is justified as follows. Player I will fold his worst hands because he is caught bluffing, and fold some of his good hands because player II will also raise with his best hands. However, if player II simply calls, he has higher chance of losing to the good hands that player I has raised with. Folding is even more disastrous as he is bluffed by the worst hands of Player I.

In both models, payoffs for player I are both a which corresponds to the initial bluff region by Player I in the first round. This fact needs to be further investigated in order to determine whether value of player I is always a allowing additional re-raises. □

4 Multiple Re-raises

The game tree allowing two raises by each player is illustrated in Figure 7, and model of more raises can extended by adding more "branches".

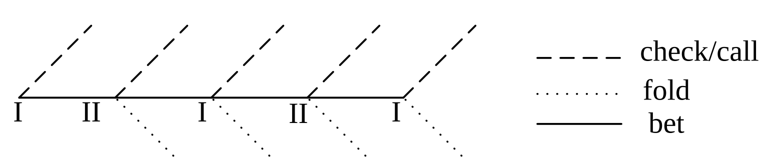

Figure 7: Game tree of two raises.

By forward induction, we conjectured the optimal strategies for both players to be as follows. For player i, $i = 1, 2$, in round j, let $f_{i,j}$ be the folding threshhold, $c_{i,j}$ the calling threshold, and $r_{i,j}$ the raising threshold. $f_{i,j} < c_{i,j} < r_{i,j} < 1$, dividing the hands into four intervals. Then player i in round j will

$$\begin{cases} \text{fold} & \text{if } x_i \in (0, f_{i,j}] \\ \text{raise} & \text{if } x_i \in (f_{i,j}, c_{i,j}] \\ \text{call/check} & \text{if } x_i \in (c_{i,j}, r_{i,j}] \\ \text{raise} & \text{if } x_i \in (r_{i,j}, 1] \end{cases}$$

$f_{I,1} = 0$ in this model because no hand will be folded by player I. Furthermore, $0 < f_{i,j}, c_{i,j}, r_{i,j} < f_{i,J}$ for all $J > j$, producing strictly smaller interval of remaining hands. Relationship of inequalities between two players' folding, calling and raising thresholds needs to be investigated. We can apply indifference conditions to each of the thresholds and solve for equilibrium conditions, but it will be complicated algebraically.

If the bet amount is restricted to be pot size, with unlimited number of bets and forbidding check-raises, Cutler solves an equilibrium using recursion [Cutler, 1975]. Recursion is feasible because each time a player pays the same ratio to win an equivalent amount when he calls.

5 Conclusion

We investigated von Neumann's model to see that bluffing is an important strategy to gain advantage. Giving player II's option to re-raise, thus option to bluff after possible bluff after player I, eliminates some of player I's advantage.

Findings in this paper may be applied to real poker. The paper shows that in optimal strategies, a player should bluff with their worst hands, but not mediocre hands, because there is slim hope of winning with worst hands. However, when one raises with a mediocre hand, the opponent is likely to call with a better hand and fold worse hands. Thus raising by the first player magnifies his loss by losing more from inferior hands and gaining none from the other player folding worse hand. In addition, if the player is getting re-raised, he would have a hard time deciding whether to fold or call with a mediocre hand, because he could falsely think that the opponent is bluffing him. However, calling would result in even bigger loss. It could have been all avoided by simply checking an OK hand in the beginning!

In addition, it has psychological implication as well. For a bad hand, getting re-raised is getting caught bluffing unfortunately, but there will not be a hard time or feeling to fold the hand. Re-raising on top of the other player, however, would be a big mistake (most of the time, unless the other player is not adhering to the other optimal strategy). Psychologically, a marginal and disastrous loss would result in a loose play which turns into even bigger loss [Smith et al., 2009].

Since unlimited number of raises are allowed in real poker, extension to more rounds need is important too. The extensions can help us to conclude about players' optimal strategies corresponding to different regions of cards, and whether player I's expected payoff from optimal strategy is always a, proportion of hands he has bluffed in initial raise. In addition, it gains insight whether there is a better equilibrium play for either player.

In addition, the paper also introduced the payoff square, which could be of important use for more complicated hand distribution for games of two or three players because of its straightforward presentation of payoffs.

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